Derivation of Smith Chart Equations

The relationship between impedance and reflection coefficient is given by:

$$Z(z) = Z_o\left[\frac{1+\Gamma(z)}{1-\Gamma(z)}\right] \qquad (1)$$

where Z_o is the characteristic impedance of the system. The normalized impedance is

$$Z_n(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)} = \frac{1+\Gamma}{1-\Gamma} \qquad (2)$$

The reflection coefficient and the normalized impedance have the form:

$$\Gamma = \Gamma_r + j\Gamma_i \qquad (3)$$

and

$$Z_n = r + jx \qquad (4)$$

Therefore

$$r + jx = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$$

$$= \frac{\left[(1+\Gamma_r) + j\Gamma_i\right]\left[(1-\Gamma_r) + j\Gamma_i\right]}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$= \frac{1-\Gamma_r^2 + j\Gamma_i(1+\Gamma_r) + j\Gamma_i(1-\Gamma_r) - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

Separating real and imaginary components,

$$r + jx = \frac{1-\Gamma_r^2 - \Gamma_i^2 - 2j\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad (5)$$

Isolating the real part from both sides

$$r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad (6)$$

Multiplying through by the denominator,

$$r\left[1+\Gamma_r^2-2\Gamma_r+\Gamma_i^2\right]=1-\Gamma_r^2-\Gamma_i$$

$$\Gamma_r^2(r+1)+\Gamma_i^2(r+1)-2r\Gamma_r=1-r$$

$$\Gamma_r^2+\Gamma_i^2-\frac{2r\Gamma_r}{1+r}=\frac{1-r}{1+r} \qquad (7)$$

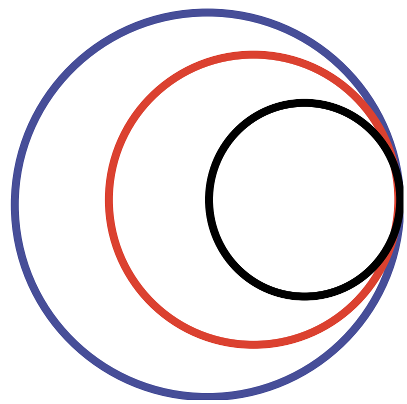
Completing the square

$$\Gamma_r^2+\Gamma_i^2-\frac{2r\Gamma_r}{1+r}+\frac{r^2}{(1+r)^2}=\frac{1-r}{1+r}+\frac{r^2}{(1+r)^2}$$

$$\left(\Gamma_r-\frac{r}{1+r}\right)^2+\Gamma_i^2=\frac{1}{(1+r)^2}$$

This is the equation of a circle centered at

$\left(\frac{r}{1+r},0\right)$ and of radius $\frac{1}{1+r}$



Equating the imaginary parts in (5)

$$x=\frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2} \qquad (8)$$

$$x\left[1+\Gamma_r^2-2\Gamma_r+\Gamma_i^2\right]=2\Gamma_i \qquad (9)$$

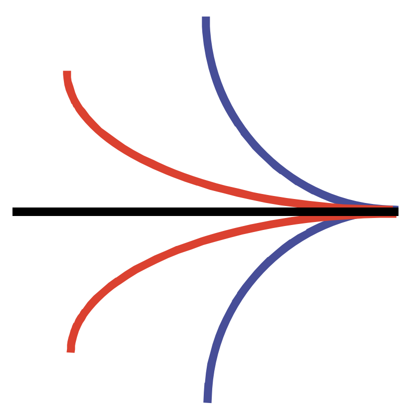
$$\Gamma_r^2x-2x\Gamma_r+x\Gamma_i^2-2\Gamma_i=-x \qquad (10)$$

$$\Gamma_r^2-2\Gamma_r+1+\Gamma_i^2-\frac{2\Gamma_i}{x}+\frac{1}{x^2}=\frac{1}{x^2}-1+1$$

$$\left(\Gamma_r-1\right)^2+\left(\Gamma_i-\frac{1}{x}\right)^2=\frac{1}{x^2}$$

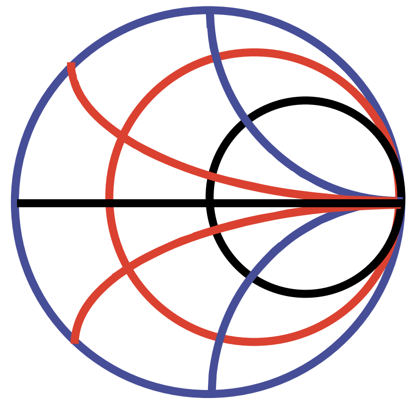
This is the equation of a circle centered at

$\left(1,\frac{1}{x}\right)$ of radius $\frac{1}{x}$



The reflection coefficient is given by

$$\Gamma = \frac{Z_n - 1}{Z_n + 1} = \frac{r - 1 + jx}{r + 1 + jx}$$



We also have

$$|\Gamma| = \left[\frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2} \leq 1$$

$$Z_n = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

$$y = \frac{1}{Z_n} = \frac{1 - \Gamma(z)}{1 + \Gamma(z)}$$

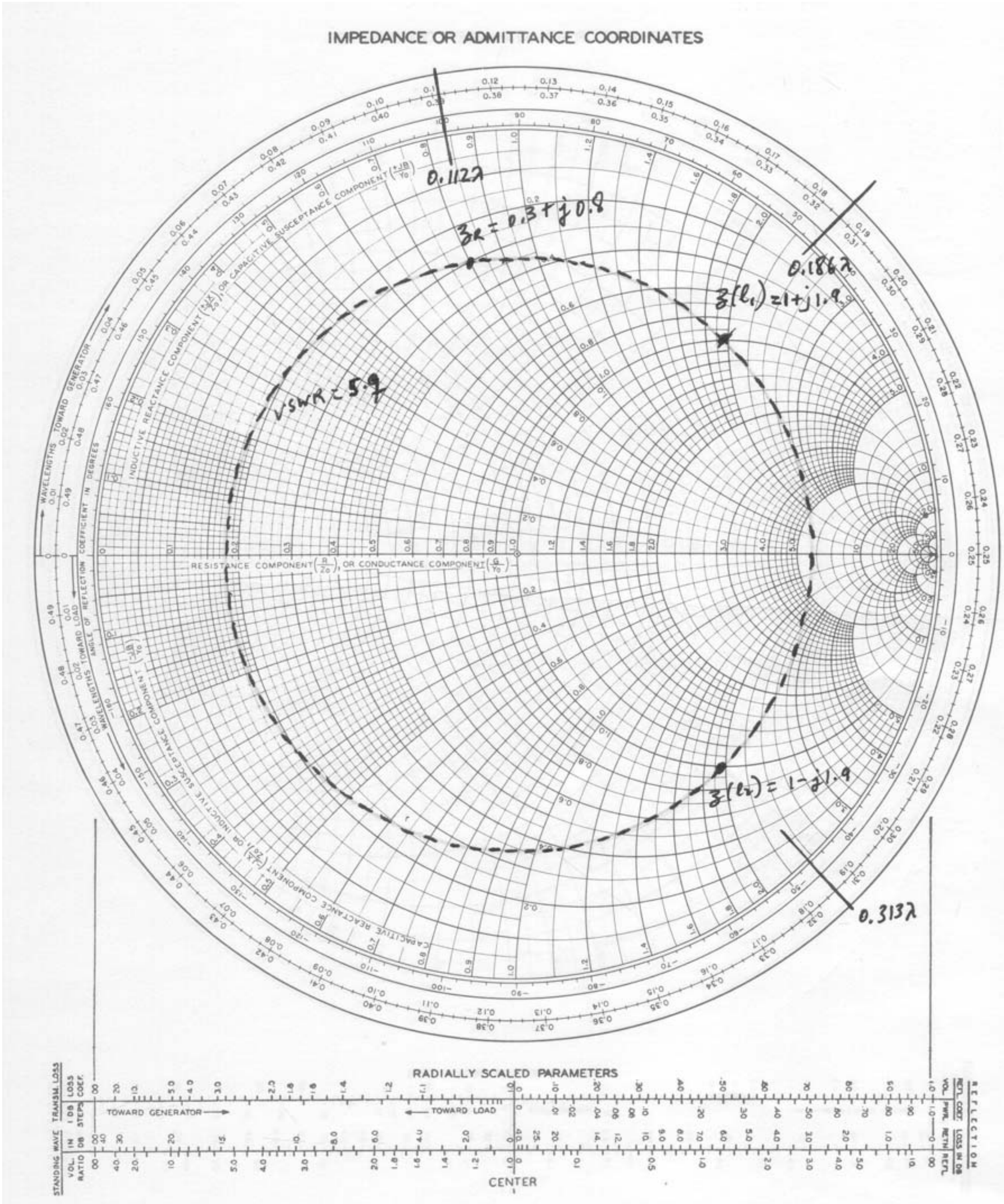
Thus, going from normalized impedance to normalized admittance corresponds to a 180 degree shift.

3 ways to move on the Smith chart

1 Constant SWR circle → displacement along transmission line

2. Constant resistance (conductance) circle → addition of reactance (susceptance)

3. Constant reactance (susceptance) arc → addition of resistance (conductance)


IMPEDANCE OR ADMITTANCE COORDINATES
0.112λ
zL = 0.3+j0.8
0.186λ
z(l1) = 1+j1.9
VSWR = 5.9
z(l2) = 1-j1.9
0.313λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE COMPONENT (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE COMPONENT (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
RADIALLY SCALED PARAMETERS
TOWARD GENERATOR
TOWARD LOAD
CENTER