PARALLEL-PLATE WAVEGUIDES

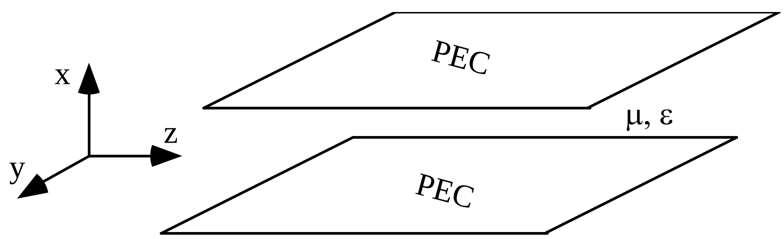
Wave Equation

$$\nabla^2 \mathbf{E} + \omega^2 \mu\varepsilon \mathbf{E} = \mathbf{0} \tag{1}$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu\varepsilon E_x \tag{2a}$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu\varepsilon E_y \tag{2b}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu\varepsilon E_z \tag{2c}$$



Transverse Electric (TE) Modes

For a parallel-plate waveguide, the plates are infinite in the y-extent; we need to study the propagation in the z-direction. The following assumptions are made in the wave equation

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

$\Rightarrow$ Assume E_y only

These two conditions define the **TE modes** and the wave equation is simplified to read

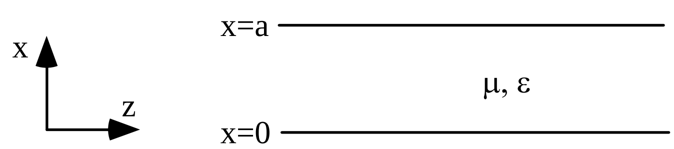
$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu\varepsilon E_y \tag{3}$$

General solution (forward traveling wave)

$$E_y(x,z) = e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x}\right] \tag{4}$$

At $x = 0$, $E_y = 0$ which leads to $A + B = 0$. Therefore, $A = -B = E_o/2j$, where E_o is an arbitrary constant

$$E_y(x,z) = E_o e^{-j\beta_z z}\sin\beta_x x \tag{5}$$



At $x = a$, $E_y(x, z) = 0$. Let a be the distance separating the two PEC plates

$$E_o e^{-j\beta_z z}\sin\beta_x a = 0 \tag{6}$$

This leads to :

$$\beta_x a = m\pi, \text{ where } m = 1, 2, 3, \ldots \tag{7}$$

or

$$\beta_x = \frac{m\pi}{a} \tag{8}$$

Moreover, from the differential equation (3), we get the *dispersion relation*

$$\beta_z^2 + \beta_x^2 = \omega^2\mu\varepsilon = \beta^2, \tag{9}$$

which leads to

$$\beta_z = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2} \tag{10}$$

where m = 1, 2, 3, ... Since propagation is to take place in the *z* direction, for the wave to propagate, we must have $\beta_z^2 > 0$, or

$$\omega^2\mu\varepsilon > \left(\frac{m\pi}{a}\right)^2 \tag{11}$$

This leads to the following *guidance condition* which will insure wave propagation

$$f > \frac{m}{2a\sqrt{\mu\varepsilon}} \tag{12}$$

The *cutoff frequency* f_c is defined to be at the onset of propagation

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} \tag{13}$$

The cutoff frequency is the frequency below which the mode associated with the index m will not propagate in the waveguide. Different modes will have different cutoff frequencies. The cutoff frequency of a mode is associated with the cutoff wavelength λ_c

$$\lambda_c = \frac{v}{f_c} = \frac{2a}{m} \tag{14}$$

Each mode is referred to as the TE_m mode. From (6), it is obvious that there is no TE_0 mode and the first TE mode is the TE_1 mode.

Magnetic Field

$$\text{From} \quad \nabla \times \mathbf{E} = -\,j\omega\mu\mathbf{H} \tag{15}$$

we have

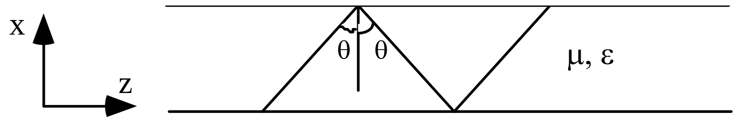
$$\mathbf{H} = \frac{-1}{j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} \tag{16}$$

which leads to

$$H_x = -\frac{\beta_z}{\omega\mu} E_o e^{-j\beta_z z} \sin \beta_x x \tag{17}$$

$$H_z = +\frac{j\beta_x}{\omega\mu} E_o e^{-j\beta_z z} \cos \beta_x x \tag{18}$$

As can be seen, there is no H_y component, therefore, the TE solution has E_y, H_x and H_z only.



From the dispersion relation, it can be shown that the propagation vector components satisfy the relations

$$\beta_Z = \beta sin\theta, \beta_X = \beta cos\theta \qquad (19)$$

where θ is the angle of incidence of the propagation vector with the normal to the conductor plates.

The phase and group velocities are given by

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{c}{\sqrt{1 - \frac{f_c^2}{f^2}}} \quad \text{and} \quad v_g = \frac{\partial\omega}{\partial\beta_z} = c\sqrt{1 - \frac{f_c^2}{f^2}}$$

The effective guide impedance is given by:

$$\eta_{TE} = \frac{E_y}{-H_x} = \frac{\eta_o}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

Transverse Magnetic (TM) modes

The magnetic field also satisfies the wave equation:

$$\nabla^2\mathbf{H} + \omega^2\mu\varepsilon\mathbf{H}=\mathbf{0} \qquad (20)$$

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} = -\omega^2\mu\varepsilon H_x \qquad (21a)$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2\mu\varepsilon H_y \qquad (21b)$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2\mu\varepsilon H_z \qquad (21c)$$

For TM modes, we assume

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

$\Rightarrow$ Assume H_y only

These two conditions define the *TM modes* and equations (21) are simplified to read

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu\varepsilon H_y \quad (22)$$

<u>General solution (forward traveling wave)</u>

$$H_y(x,z) = e^{-j\beta_z z}\left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \quad (23)$$

From $\nabla \times \mathbf{H} = -j\omega\varepsilon\mathbf{E}$ (24)

we get

$$\mathbf{E} = \frac{1}{j\omega\varepsilon} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} \quad (25)$$

This leads to

$$E_x(x,z) = \frac{\beta_z}{\omega\varepsilon} e^{-j\beta_z z}\left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \quad (26)$$

$$E_z(x,z) = \frac{\beta_x}{\omega\varepsilon} e^{-j\beta_z z}\left[-A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \quad (27)$$

At $x=0$, $E_z = 0$ which leads to $A = B = H_o/2$ where H_o is an arbitrary constant. This leads to

$$H_y(x,z) = H_o e^{-j\beta_z z} \cos \beta_x x \quad (28)$$

$$E_x(x,z) = \frac{\beta_z}{\omega\varepsilon} H_o e^{-j\beta_z z} \cos \beta_x x \quad (29)$$

$$E_z(x,z) = \frac{j\beta_x}{\omega\varepsilon} H_o e^{-j\beta_z z} \sin \beta_x x \quad (30)$$

At $x = a$, $E_z = 0$ which leads to

$$\beta_x a = m\pi, \text{ where } m = 0, 1, 2, 3, \ldots \tag{31}$$

This defines the TM modes which have only H_y, E_x and E_z components.

The effective guide impedance is given by:

$$\eta_{TM} = \frac{E_x}{H_y} = \eta_o\sqrt{1 - \frac{f_c^2}{f^2}}$$

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A PARALLEL-PLATE WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

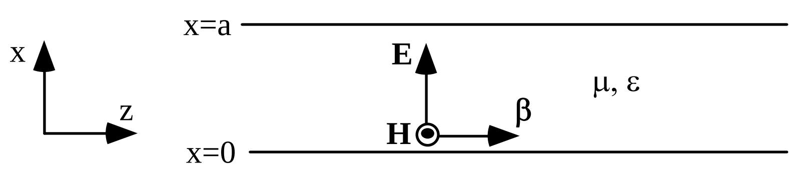
Equation (31) defines the **TM modes**; each mode is referred to as the TM_m mode. It can be seen from (28) that m=0 is a valid choice; it is called the TM_0, or *transverse electromagnetic* or TEM mode. For this mode β_x=0 and,

$$H_y = H_o e^{-j\beta_z z} \tag{32}$$

$$E_x = \frac{\beta_z}{\omega\varepsilon} H_o e^{-j\beta_z z} = \sqrt{\frac{\mu}{\varepsilon}} H_o e^{-j\beta_z z} \tag{33}$$

$$E_z = 0 \tag{34}$$

where $\beta_z = \beta$, and in which there are no x variations of the fields within the waveguide. The TEM mode has a cutoff frequency at DC and is always present in the waveguide.



TEM mode

Time-Average Poynting Vector

<u>TE modes</u>

$$\langle \mathbf{P} \rangle = \frac{1}{2}\text{Re}\{\mathbf{E} \times \mathbf{H}^*\} \tag{35}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2} \mathrm{Re}\left\{ \hat{\mathbf{y}} E_y \times \left[\hat{\mathbf{x}} H_x^* + \hat{\mathbf{z}} H_z^* \right] \right\} \tag{36}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2} \mathrm{Re}\left\{ \hat{\mathbf{z}} \frac{|E_o|^2}{\omega\mu} \beta_z \sin^2 \beta_x x + \hat{\mathbf{x}} j \frac{|E_o|^2}{\omega\mu} \beta_x \cos \beta_x x \sin \beta_x x \right\} \tag{37}$$

$$\langle \mathbf{P} \rangle = \hat{\mathbf{z}} \frac{|E_o|^2}{2\omega\mu} \beta_z \sin^2 \beta_x x \tag{38}$$

<u>TM modes</u>

$$\langle \mathbf{P} \rangle = \frac{1}{2} \mathrm{Re}\left\{ \mathbf{E} \times \mathbf{H}^* \right\} \tag{39}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2} \mathrm{Re}\left\{ \left[\hat{\mathbf{x}} E_x + \hat{\mathbf{z}} E_z \right] \times \hat{\mathbf{y}} H_y^* \right\} \tag{40}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2} \mathrm{Re}\left\{ \hat{\mathbf{z}} \frac{|H_o|^2}{\omega\varepsilon} \beta_z \cos^2 \beta_x x - \hat{\mathbf{x}} j \frac{|H_o|^2}{\omega\varepsilon} \beta_x \sin \beta_x x \cos \beta_x x \right\} \tag{41}$$

$$\langle \mathbf{P} \rangle = \hat{\mathbf{z}} \frac{|H_o|^2}{2\omega\varepsilon} \beta_z \cos^2 \beta_x x \tag{42}$$

The total time-average power is found by integrating <**P**> over the area of interest.