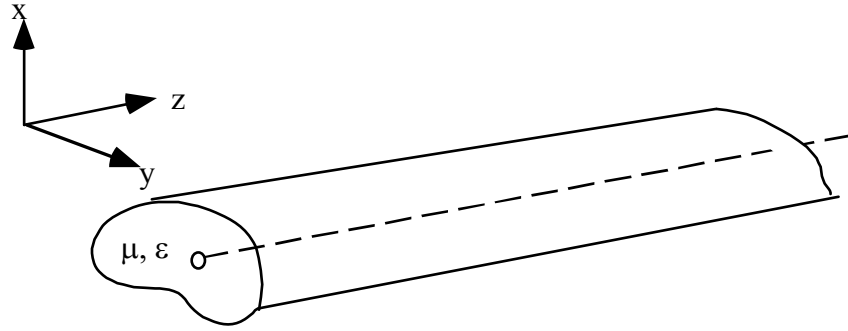


WAVEGUIDES



Maxwell's Equation

$$\nabla^2 \mathbf{E} + \omega^2 \mu \epsilon \mathbf{E} = \mathbf{0} \quad (\text{A})$$

$$\nabla^2 \mathbf{H} + \omega^2 \mu \epsilon \mathbf{H} = \mathbf{0} \quad (\text{B})$$

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set $E_z = 0$. This defines the TE modes.

From $\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \Rightarrow +j\beta_z E_y = -j\omega\mu H_x \quad (1)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \Rightarrow -j\beta_z E_x = -j\omega\mu H_y \quad (2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad (3)$$

From $\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}$, we get

$$j\omega\epsilon \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\epsilon E_x \Rightarrow \frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega\epsilon E_x \quad (4)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \Rightarrow -j\beta_z H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \quad (5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad (6)$$

We want to express all quantities in terms of H_z .

From (2), we have

$$H_y = \frac{\beta_z E_x}{\omega\mu} \quad (7)$$

in (4)

$$\frac{\partial H_z}{\partial y} + j\beta_z^2 \frac{E_x}{\omega\mu} = j\omega\epsilon E_x \quad (8)$$

Solving for E_x

$$E_x = \frac{j\omega\mu}{\beta_z^2 - \omega^2\mu\epsilon} \frac{\partial H_z}{\partial y} \quad (9)$$

From (1)

$$H_x = \frac{-\beta_z E_y}{\omega\mu} \quad (10)$$

in (5)

$$+j \frac{\beta_z^2 E_y}{\omega\mu} - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \quad (11)$$

so that

$$E_y = \frac{-j\omega\mu}{\beta_z^2 - \omega^2\mu\epsilon} \frac{\partial H_z}{\partial x} \quad (12)$$

$$H_x = \frac{j\beta_z}{\beta_z^2 - \omega^2\mu\epsilon} \frac{\partial H_z}{\partial x} \quad (13)$$

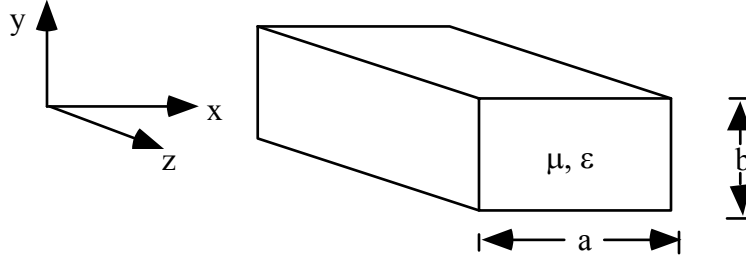
$$H_y = \frac{j\beta_z}{\beta_z^2 - \omega^2\mu\epsilon} \frac{\partial H_z}{\partial y} \quad (14)$$

$$E_z = 0 \quad (15)$$

Combining solutions for E_x and E_y into (3) gives

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = [\beta_z^2 - \omega^2 \mu \epsilon] H_z \quad (16)$$

RECTANGULAR WAVEGUIDES



If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

TE Modes

The general solution for TE modes with $E_z=0$ is obtained from (16)

$$H_z = e^{-j\beta_z z} \left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \left[C e^{-j\beta_y y} + D e^{+j\beta_y y} \right] \quad (17)$$

$$E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \epsilon} e^{-j\beta_z z} \left[-A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \left[C e^{-j\beta_y y} + D e^{+j\beta_y y} \right] \quad (18)$$

$$E_x = \frac{j\beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \epsilon} e^{-j\beta_z z} \left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \left[-C e^{-j\beta_y y} + D e^{+j\beta_y y} \right] \quad (19)$$

At $y=0$, $E_x=0$ which leads to $C=D$

At $x=0$ $E_y=0$ which leads to $A=B$

$$H_z = H_0 e^{-j\beta_z z} \cos \beta_x x \cos \beta_y y \quad (20)$$

$$E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \epsilon} H_0 e^{-j\beta_z z} \sin \beta_x x \cos \beta_y y \quad (21)$$

$$E_x = \frac{j\beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \epsilon} H_0 e^{-j\beta_z z} \cos \beta_x x \sin \beta_y y \quad (22)$$

At $x=a$, $E_y=0$; this leads to $\beta_x = \frac{m\pi}{a}$

At $y=b$, $E_x=0$; this leads to $\beta_y = \frac{n\pi}{b}$

The dispersion relation is obtained by placing (20) in (16)

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2 \mu \epsilon \quad (23)$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \omega^2 \mu \epsilon \quad (24)$$

and

$$\beta_z = \sqrt{\omega^2 \mu \epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad (25)$$

The guidance condition is

$$\omega^2 \mu \epsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad (26)$$

or

$f > f_c$ where f_c is the cutoff frequency of the TE_{mn} mode given by the relation

$$f_c = \frac{1}{2\sqrt{\mu \epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad (27)$$

The TE_{mn} mode will not propagate unless f is greater than f_c . Obviously, different modes will have different cutoff frequencies.

TM Modes

The transverse magnetic modes for a general waveguide are obtained by assuming $H_z = 0$. By duality with the TE modes, we have

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = [\beta_z^2 - \omega^2 \mu \epsilon] E_z \quad (28)$$

with general solution

$$E_z = e^{-j\beta_z z} \left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right] \left[C e^{-j\beta_y y} + D e^{+j\beta_y y} \right] \quad (29)$$

The boundary conditions are

At $x=0$, $E_z=0$ which leads to $A=-B$

At $y=0$, $E_z=0$ which leads to $C=-D$

At $x=a$, $E_z=0$ which leads to $\beta_x = \frac{m\pi}{a}$

At $y=b$, $E_z=0$ which leads to $\beta_y = \frac{n\pi}{b}$

so that the generating equation for the TM_{mn} modes is

$$E_z = E_0 e^{-j\beta_z z} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (30)$$

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

For additional information on the field equations see Rao (6th Edition), page 607 Table 9.1.

There is no TE_{00} mode

There are no TM_{m0} or TM_{0n} modes

The first TE mode is the TE_{10} mode

The first TM mode is the TM_{11} mode

Impedance of a Waveguide

For a TE mode, we define the transverse impedance as

$$\eta_{gTE} = \frac{-E_y}{H_x} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta_z}$$

From the relationship for β_z and using

$$f_c^2 = \frac{1}{4\pi^2\mu\epsilon} \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right] \quad (31)$$

we get

$$\eta_{gTE} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}} \quad (32)$$

where η is the intrinsic impedance $\eta = \sqrt{\frac{\mu}{\epsilon}}$. Analogously, for TM modes, it can be shown that

$$\eta_{gTM} = \eta \sqrt{1 - \frac{f_c^2}{f^2}} \quad (33)$$

Power Flow in a Rectangular Waveguide (TE₁₀)

The time-average Poynting vector for the TE₁₀ mode in a rectangular waveguide is given by

$$\langle \mathbf{P} \rangle = \frac{1}{2} \text{Re}[\mathbf{E} \times \mathbf{H}^*] = \hat{\mathbf{z}} \frac{|E_0|^2}{2} \frac{\beta_z}{\omega\mu} \sin^2 \frac{\pi x}{a} \quad (34)$$

$$\langle \text{Power} \rangle = \int_0^a \int_0^b \frac{|E_0|^2}{2} \frac{\beta_z}{\omega\mu} \sin^2 \frac{\pi x}{a} dx dy \quad (35)$$

$$\langle \text{Power} \rangle = \frac{|E_0|^2}{4} \frac{\beta_z ab}{\omega\mu} = \frac{|E_0|^2 ab}{4\eta_{gTE_{10}}} \quad (36)$$

Therefore the time-average power flow in a waveguide is proportional to its cross-section area.