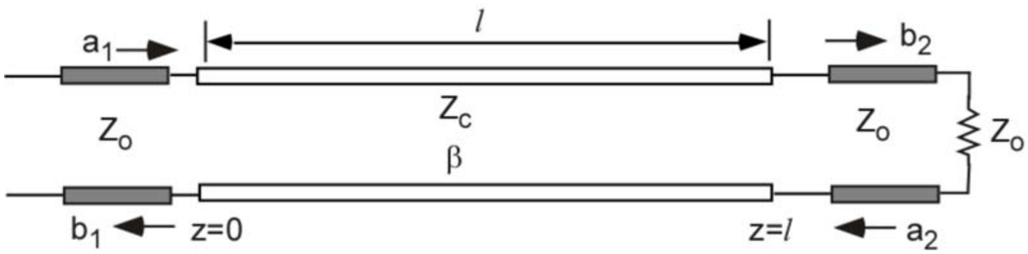
Homework 3 - Solutions

Problem 1



(a) Calculating *S11*

At the load *(z=l)*, the reflection coefficient is: $\Gamma_R = \dfrac{Z_o - Z_c}{Z_o + Z_c} = -\Gamma$

At the input of the transmission line *(z=0)*, the reflection coefficient is Γ_{in}

$$\Gamma_{in} = \Gamma_R e^{-2j\beta l} = -\Gamma X^2 \text{ with } X = e^{-j\beta l}$$

The input impedance is:

$$Z_{in} = Z_c \left[\frac{1 - \Gamma X^2}{1 + \Gamma X^2} \right]$$

S11 is the reflection coefficient at the input with respect to reference line (with characteristic impedance

Zo)

$$S_{11} = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} = \frac{Z_c \left[\dfrac{1 - \Gamma X^2}{1 + \Gamma X^2} \right] - Z_o}{Z_c \left[\dfrac{1 - \Gamma X^2}{1 + \Gamma X^2} \right] - Z_o} = \frac{-\Gamma X^2 \left[Z_c + Z_o \right] + Z_c - Z_o}{-\Gamma X^2 \left[Z_c - Z_o \right] + Z_c + Z_o}$$

$$S_{11} = \frac{\Gamma - \Gamma X^2}{1 - \Gamma^2 X^2} = \frac{(1 - X^2)\Gamma}{1 - \Gamma^2 X^2}$$

(b) Calculating *S21*

for a test line, we have

$$V_t(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z}$$

$$I_t(z)=\frac{1}{Z_c}\left[V_+e^{-j\beta z}-V_-e^{+j\beta z}\right]$$

Let us define:

$$\Gamma=\frac{Z_c-Z_o}{Z_c+Z_o}$$

$$X=e^{-j\beta l}$$

We also define

$$A_1=a_1\sqrt{Z_o}\ ,\quad B_1=b_1\sqrt{Z_o}$$

By definition S_{21} is given by:

$$S_{21}=\frac{b_2}{a_1}=\frac{V_t(l)}{a_1\sqrt{Z_o}}=\frac{V_t(l)}{A_1}$$

$$V_t(z)=V_+e^{-j\beta z}\left[1+\frac{V_-}{V_+}e^{+2j\beta z}\right]$$

$$\frac{V_-}{V_+}=\Gamma_t(0)\Rightarrow V_t(z)=V_+e^{-j\beta z}\left[1+\Gamma_t(0)e^{+2j\beta z}\right]$$

$$\Gamma_t(0)=\Gamma_t(l)e^{-2j\beta l}=\Gamma_t(l)X^2=-\Gamma X^2$$

so that $\quad V_t(l)=V_+e^{-j\beta l}\left[1+\Gamma_t(0)e^{+2j\beta l}\right]=V_+X\left[1-\Gamma X^2X^{-2}\right]=V_+X\left[1-\Gamma\right]\qquad (1)$

We now need to find V_+

At the junction between the reference and the test line, the voltage and the current must be continuous. This gives:

$$A_1+B_1=V_++V_-$$

and

$$\frac{1}{Z_o}\left[A_1-B_1\right]=\frac{1}{Z_c}\left[V_+-V_-\right]$$

from which

$$A_1+B_1=V_+\left[1+\Gamma_t(0)\right]=V_+\left[1-\Gamma X^2\right]\qquad (2)$$

$$A_1-B_1=\frac{Z_o}{Z_c}V_+\left[1-\Gamma_t(0)\right]=\frac{Z_o}{Z_c}V_+\left[1+\Gamma X^2\right]\qquad (3)$$

Adding (2) and (3)

$$2A_1 = V_+\left[1-\Gamma X^2 + \frac{Z_o}{Z_c}\left(1+\Gamma X^2\right)\right]$$

Extracting V_+, *we get:*

$$V_+ = \frac{2A_1}{\left[1-\Gamma X^2 + \frac{Z_o}{Z_c}\left(1+\Gamma X^2\right)\right]} = \frac{2A_1}{\left[1-\Gamma X^2\Gamma\right]\left(1+\frac{Z_o}{Z_c}\right)} = \frac{2A_1}{\left[1+\frac{Z_o}{Z_c}\right]}\frac{1}{1-\Gamma^2X^2} \qquad (4)$$

Since $\frac{2}{\left[1+\frac{Z_o}{Z_c}\right]} = 1+\Gamma$, substituting in (4) gives $V_+ = \frac{\left[1+\Gamma\right]A_1}{1-\Gamma^2X^2}$

Substituting for $V+$ in (1) gives $V_t(l) = \frac{A_1\left[1+\Gamma\right]\left[1-\Gamma\right]X}{1-\Gamma^2X^2} = \frac{A_1\left[1-\Gamma^2\right]X}{1-\Gamma^2X^2}$

so that $S_{21} = \frac{V_t(l)}{A_1} = \frac{\left(1-\Gamma^2\right)X}{1-\Gamma^2X^2}$

In conclusion:

$S_{11} = \frac{\left(1-X^2\right)\Gamma}{1-\Gamma^2X^2}$ and $S_{21} = \frac{\left(1-\Gamma^2\right)X}{1-\Gamma^2X^2}$

Problem 2

(a) The network is reciprocal since $S_{ij}=S_{ji}$ for all i, j

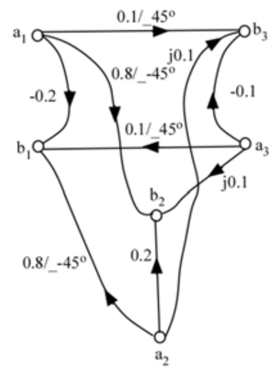
(b) Remember that S_{ij}^2 is the fraction of power leaving port j for a unit of power input to port i. The criterion for a network with N ports to be lossless is $\sum_{j=1}^{N}\left|S_{ji}\right|^2 = 1$ for all i

(c)

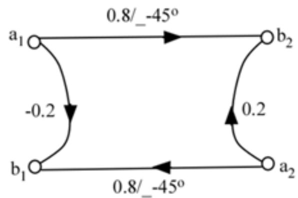
Port 1: $\left|S_{11}\right|^2 + \left|S_{21}\right|^2 + \left|S_{31}\right|^2 = 0.04 + 0.64 + 0.01 = 0.69 < 1$

The 3-port is lossy

(d)

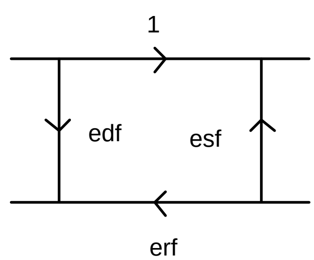


(e) With a 50 Ω load at port 3, there is no input at a_3. So, the SFG in (d) simplifies to

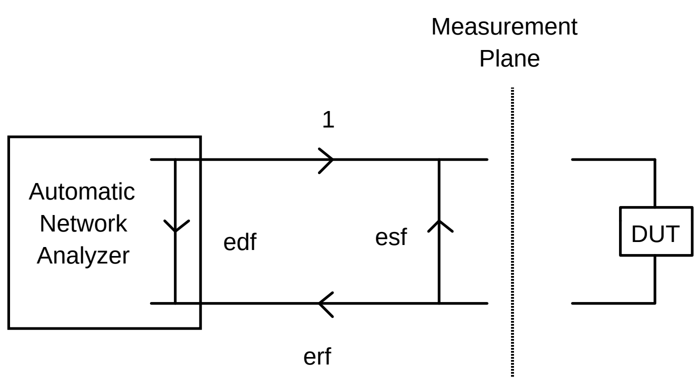


$$S:\begin{bmatrix} -0.2 & 0.8\angle-45^\circ \\ 0.8\angle-45^\circ & 0.2 \end{bmatrix}$$

Problem 3



(a)



(b) $S_{11m} = E_{DF} + \dfrac{E_{RF}S_{11a}}{1 - S_{11a}E_{SF}}$

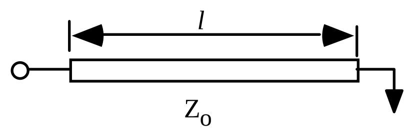
It is best to solve part (b) first, and then use part (a) as a special case.

$$\left.\begin{array}{c} \text{Matched termination} \\ \text{Offset short} \\ \text{Shielded open} \end{array}\right\} \text{Cal standards}$$

(1) Matched termination $\Rightarrow S_{11a} = 0$

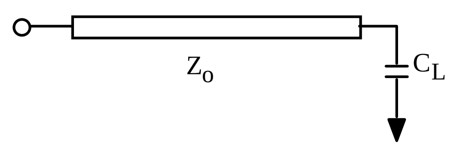
$$S_{11m}(match) = A = E_{DF} \quad (1)$$

(2) Offset short $\Rightarrow S_{11a} = e^{j\theta} = x$



$$\theta = \pi\left(1 - \frac{4l}{\lambda}\right)$$

$$S_{11m}(offset\ short) = E_{DF} + \frac{E_{RF}x}{1 - xE_{SF}} = B \qquad (2)$$



$$S_{11a} = e^{j\beta} = y$$

$$\beta = -2\tan^{-1}\left(\omega C_L Z_o\right)$$

$$S_{11m}(shielded\ open) = E_{DF} + \frac{E_{RF}y}{1 - yE_{SF}} = C \qquad (3)$$

Combining (2) and (3) with (1), we get

$$\begin{bmatrix} x(B-A) & x \\ y(C-A) & y \end{bmatrix}\begin{bmatrix} E_{SF} \\ E_{RF} \end{bmatrix} = \begin{bmatrix} B-A \\ C-A \end{bmatrix}$$

From which

$$E_{SF} = \frac{y(B-A) - x(C-A)}{xy(B-C)}, \quad E_{RF} = \frac{(B-A)(C-A)(x-y)}{xy(B-C)}, \quad E_{DF} = A$$

For part (a) set $C_L = 0 \Rightarrow \beta = 0 \Rightarrow S_{11a}(open) = +1 \Rightarrow y = +1$

$$E_{DF} = A,\ E_{SF} = \frac{(B-A) - x(C-A)}{x(B-C)},\ E_{RF} = \frac{(B-A)(C-A)(x-1)}{x(B-C)}$$