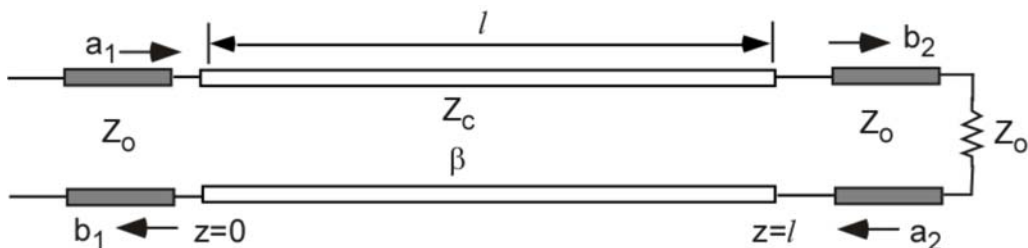


Homework 3 - Solutions

Problem 1



(a) Calculating S_{11}

At the load ($z=l$), the reflection coefficient is: $\Gamma_R = \frac{Z_o - Z_c}{Z_o + Z_c} = -\Gamma$

At the input of the transmission line ($z=0$), the reflection coefficient is Γ_{in}

$$\Gamma_{in} = \Gamma_R e^{-2j\beta l} = -\Gamma X^2 \text{ with } X = e^{-j\beta l}$$

The input impedance is:

$$Z_{in} = Z_c \left[\frac{1 - \Gamma X^2}{1 + \Gamma X^2} \right]$$

S_{11} is the reflection coefficient at the input with respect to reference line (with characteristic impedance

Z_o)

$$S_{11} = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} = \frac{Z_c \left[\frac{1 - \Gamma X^2}{1 + \Gamma X^2} \right] - Z_o}{Z_c \left[\frac{1 - \Gamma X^2}{1 + \Gamma X^2} \right] + Z_o} = \frac{-\Gamma X^2 [Z_c + Z_o] + Z_c - Z_o}{-\Gamma X^2 [Z_c - Z_o] + Z_c + Z_o}$$

$$S_{11} = \frac{\Gamma - \Gamma X^2}{1 - \Gamma^2 X^2} = \frac{(1 - X^2)\Gamma}{1 - \Gamma^2 X^2}$$

(b) Calculating S_{21}

for a test line, we have

$$V_t(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z}$$

$$I_t(z) = \frac{I}{Z_c} [V_+ e^{-j\beta z} - V_- e^{+j\beta z}]$$

Let us define:

$$\Gamma = \frac{Z_c - Z_o}{Z_c + Z_o}$$

$$X = e^{-j\beta l}$$

We also define

$$A_l = a_l \sqrt{Z_o}, \quad B_l = b_l \sqrt{Z_o}$$

By definition S_{2l} is given by:

$$S_{2l} = \frac{b_2}{a_1} = \frac{V_t(l)}{a_1 \sqrt{Z_o}} = \frac{V_t(l)}{A_l}$$

$$V_t(z) = V_+ e^{-j\beta z} \left[1 + \frac{V_-}{V_+} e^{+2j\beta z} \right]$$

$$\frac{V_-}{V_+} = \Gamma_t(0) \Rightarrow V_t(z) = V_+ e^{-j\beta z} [1 + \Gamma_t(0) e^{+2j\beta z}]$$

$$\Gamma_t(0) = \Gamma_t(l) e^{-2j\beta l} = \Gamma_t(l) X^2 = -\Gamma X^2$$

$$\text{so that } V_t(l) = V_+ e^{-j\beta l} [1 + \Gamma_t(0) e^{+2j\beta l}] = V_+ X [1 - \Gamma X^2 X^{-2}] = V_+ X [1 - \Gamma] \quad (1)$$

We now need to find V_+

At the junction between the reference and the test line, the voltage and the current must be continuous.

This gives:

$$A_l + B_l = V_+ + V_-$$

and

$$\frac{I}{Z_o} [A_l - B_l] = \frac{I}{Z_c} [V_+ - V_-]$$

from which

$$A_l + B_l = V_+ [1 + \Gamma_t(0)] = V_+ [1 - \Gamma X^2] \quad (2)$$

$$A_l - B_l = \frac{Z_o}{Z_c} V_+ [1 - \Gamma_t(0)] = \frac{Z_o}{Z_c} V_+ [1 + \Gamma X^2] \quad (3)$$

Adding (2) and (3)

$$2A_l = V_+ \left[1 - \Gamma X^2 + \frac{Z_o}{Z_c} (1 + \Gamma X^2) \right]$$

Extracting V_+ , we get:

$$V_+ = \frac{2A_l}{\left[1 - \Gamma X^2 + \frac{Z_o}{Z_c} (1 + \Gamma X^2) \right]} = \frac{2A_l}{\left[1 - \Gamma X^2 \Gamma \right] \left(1 + \frac{Z_o}{Z_c} \right)} = \frac{2A_l}{\left[1 + \frac{Z_o}{Z_c} \right]} \frac{1}{1 - \Gamma^2 X^2} \quad (4)$$

Since $\frac{2}{\left[1 + \frac{Z_o}{Z_c} \right]} = 1 + \Gamma$, substituting in (4) gives $V_+ = \frac{[1 + \Gamma] A_l}{1 - \Gamma^2 X^2}$

Substituting for V_+ in (1) gives $V_t(l) = \frac{A_l [1 + \Gamma] [1 - \Gamma] X}{1 - \Gamma^2 X^2} = \frac{A_l [1 - \Gamma^2] X}{1 - \Gamma^2 X^2}$

so that $S_{2l} = \frac{V_t(l)}{A_l} = \frac{(1 - \Gamma^2) X}{1 - \Gamma^2 X^2}$

In conclusion:

$$S_{1l} = \frac{(1 - X^2) \Gamma}{1 - \Gamma^2 X^2} \text{ and } S_{2l} = \frac{(1 - \Gamma^2) X}{1 - \Gamma^2 X^2}$$

Problem 2

(a) The network is reciprocal since $S_{ij} = S_{ji}$ for all i, j

(b) Remember that S_{ij}^2 is the fraction of power leaving port j for a unit of power input to port i . The

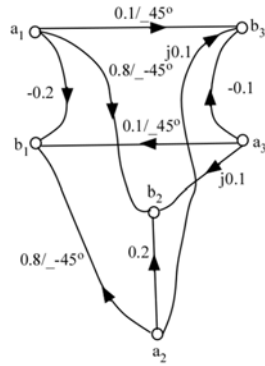
criterion for a network with N ports to be lossless is $\sum_{j=1}^N |S_{ji}|^2 = 1$ for all i

(c)

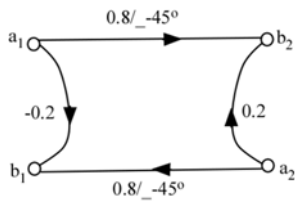
Port 1: $|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 0.04 + 0.64 + 0.01 = 0.69 < 1$

The 3-port is lossy

(d)

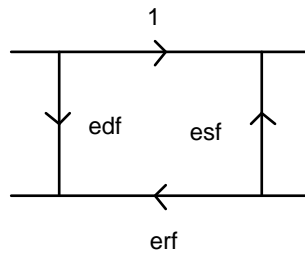


(e) With a 50Ω load at port 3, there is no input at a_3 . So, the SFG in (d) simplifies to

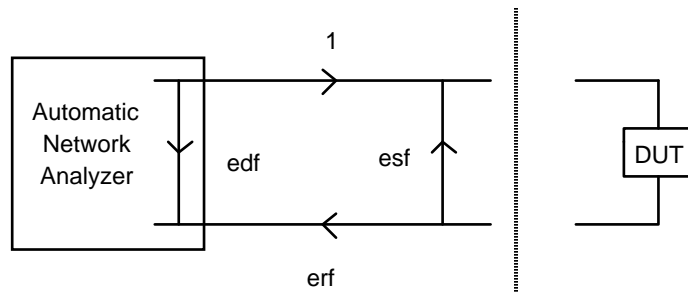


$$S : \begin{bmatrix} -0.2 & 0.8 \angle -45^\circ \\ 0.8 \angle -45^\circ & 0.2 \end{bmatrix}$$

Problem 3



Measurement Plane



(a)

$$(b) S_{11m} = E_{DF} + \frac{E_{RF} S_{11a}}{1 - S_{11a} E_{SF}}$$

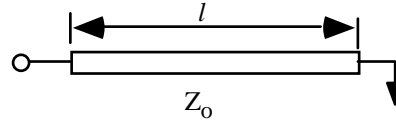
It is best to solve part (b) first, and then use part (a) as a special case.

$\left. \begin{array}{l} \text{Matched termination} \\ \text{Offset short} \\ \text{Shielded open} \end{array} \right\} \text{Cal standards}$

(1) Matched termination $\Rightarrow S_{11a} = 0$

$$S_{11m}(\text{match}) = A = E_{DF} \quad (1)$$

(2) Offset short $\Rightarrow S_{11a} = e^{j\theta} = x$



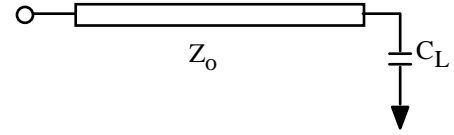
$$\theta = \pi \left(1 - \frac{4l}{\lambda} \right)$$

$$S_{11m}(\text{offset short}) = E_{DF} + \frac{E_{RF}x}{1 - xE_{SF}} = B \quad (2)$$

$$S_{11a} = e^{j\beta} = y$$

$$\beta = -2 \tan^{-1}(\omega C_L Z_0)$$

$$S_{11m}(\text{shielded open}) = E_{DF} + \frac{E_{RF}y}{1 - yE_{SF}} = C \quad (3)$$



Combining (2) and (3) with (1), we get

$$\begin{bmatrix} x(B-A) & x \\ y(C-A) & y \end{bmatrix} \begin{bmatrix} E_{SF} \\ E_{RF} \end{bmatrix} = \begin{bmatrix} B-A \\ C-A \end{bmatrix}$$

From which

$$E_{SF} = \frac{y(B-A) - x(C-A)}{xy(B-C)}, \quad E_{RF} = \frac{(B-A)(C-A)(x-y)}{xy(B-C)}, \quad E_{DF} = A$$

For part (a) set $C_L = 0 \Rightarrow \beta = 0 \Rightarrow S_{11a}(\text{open}) = +1 \Rightarrow y = +1$

$$E_{DF} = A, \quad E_{SF} = \frac{(B-A) - x(C-A)}{x(B-C)}, \quad E_{RF} = \frac{(B-A)(C-A)(x-1)}{x(B-C)}$$