ECE 451 HOMEWORK # 6 Due Wednesday, April 23, 2025

Causality assessment in S-parameter with Hilbert transform

1. Introduction

Frequency-domain measurement of network parameters is an incredibly powerful technique. When the networks are linear and time-invariant, frequency-domain measurements can completely characterize these systems. From these measurements, the time-domain behavior of these systems can be predicted accurately through the use of transform techniques.

For actual networks, the time-domain signals must exhibit certain properties that are associated with a physical system. For instance, a signal in the time domain must be real, stable and causal. Violation of these properties would lead to non-physical situations. These properties lead to some important constraints to the time-domain signals and their associated frequency-domain responses. This homework will examine the causality property of a physical system from its frequency-domain measurements of scattering parameters. The subject of the passivity, stability, and causality of scattering parameter models is a very rich field. There is a basic but excellent paper on the topic at: http://ieeexplore.ieee.org/document/4358038/

A. Time-Domain Response

The time-domain response of a network must be real. From transform theory, we know that this correlates to the relationship

$$V(-f) = V^{*}(f) \Leftrightarrow v(t)\ real \tag{1}$$

Since negative frequencies are not measured in a network analyzer, this has limited relevance to measurements; however, the relationship must be applied in determining the time-domain response from the measurement data. When we use S-parameters in a transient simulation, the simulator uses the above property of Hermitian symmetry to reproduce the negative frequencies and perform an inverse Fourier transform to generate an impulse response for the S-parameter measurements. The simulator then performs discrete convolution on this impulse response with whichever stimulus you provided to generate the output.

B. Causality

Time-domain signals associated with physical networks must be causal. This means that the signal is null for t<0. In other words,

$$h(t)=0,\ t<0;$$

The time-domain function can be considered as the superposition of an even function and an odd function:

$$h_e(t) = \frac{1}{2}\left[h(t) + h(-t)\right] \tag{7}$$

$$h_o(t) = \frac{1}{2}\left[h(t) - h(-t)\right] \tag{8}$$

Also, it can be shown that the transform of the transfer function is given by:

$$H(f) = H_e(f) - j\hat{H}_e(f) \tag{9}$$

and

$$H(f) = \hat{H}_o(f) + jH_o(f) \tag{9}$$

where $H(f)$, $H_e(f)$, and $H_o(f)$ are the Fourier transforms of $h(t)$, $h_e(t)$, and $h_o(t)$ respectively and $\hat{H}_e(f)$ and $\hat{H}_o(f)$ are the Hilbert transforms of $H_e(f)$ and $H_o(f)$. The Hilbert transform relationship is given by:

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau} d\tau \tag{10}$$

From this we see that in order for causality to exist, the imaginary part of the transform must be related to the real part through the Hilbert transform and vice versa. When dealing with discrete data points, the discrete version of the Hilbert transform relationship is more appropriate. It is given by [1]:

$$\hat{f}_k = \begin{cases} \dfrac{2}{\pi} \displaystyle\sum_{n \text{ odd}} \frac{f_n}{k-n}, & k \text{ even} \\ \dfrac{2}{\pi} \displaystyle\sum_{n \text{ even}} \frac{f_n}{k-n}, & k \text{ odd} \end{cases} \tag{11}$$

So now your discrete S-parameter measurement (say S_{21}) is a function of frequency and can be written as a vector.

Your S-parameter files are frequency-domain measurements, so in these equations, f_n is the discrete frequency data point that you have from your S parameters. $\hat{f}_n$ is the Hilbert transform point.

2. Verifying causality in measured S-parameters

You are given a measurement data of a passive 2-port network, *allfreq.s2p*. Write a script that calculates the discrete Hilbert transform as in (11). Use the script to:

a) calculate the Hilbert transform of Re(S_{21}). Compare it with Im(S_{21}).
b) calculate the Hilbert transform of Im(S_{21}). Compare it with Re(S_{21}).

References

[1] S. C. Kak, "The discrete Hilbert transform," in *Proceedings of the IEEE*, vol. 58, no. 4, pp. 585-586, April 1970, doi: 10.1109/PROC.1970.7696.