ECE 451
Advanced Microwave Measurements

3. The Smith Chart

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Derivation of the Smith Chart

The relationship between impedance and reflection coefficient is given by:

$$Z(z) = Z_o\left[\frac{1+\Gamma(z)}{1-\Gamma(z)}\right]$$

where Z_o is the characteristic impedance of the system. The normalized impedance is

$$Z_n(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)} = \frac{1+\Gamma}{1-\Gamma}$$

The reflection coefficient and the normalized impedance have the form:

$$\Gamma = \Gamma_r + j\Gamma_i \quad \text{and} \quad Z_n = r + jx$$

Derivation of the Smith Chart

Therefore

$$r + jx = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i} = \frac{\left[(1+\Gamma_r) + j\Gamma_i\right]\left[(1-\Gamma_r) + j\Gamma_i\right]}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

Separating real and imaginary components,

$$r + jx = \frac{1-\Gamma_r^2 + j\Gamma_i(1+\Gamma_r) + j\Gamma_i(1-\Gamma_r) - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

Isolating the real part from both sides

$$r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

Derivation of the Smith Chart

Multiplying through by the denominator,

$$r\left[1+\Gamma_r^2-2\Gamma_r+\Gamma_i^2\right]=1-\Gamma_r^2-\Gamma_i$$

$$\Gamma_r^2(r+1)+\Gamma_i^2(r+1)-2r\Gamma_r=1-r$$

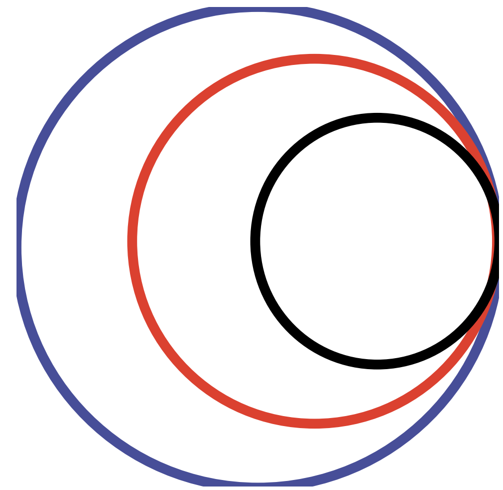
$$\Gamma_r^2+\Gamma_i^2-\frac{2r\Gamma_r}{1+r}+\frac{r^2}{(1+r)^2}=\frac{1-r}{1+r}+\frac{r^2}{(1+r)^2}$$

Completing the square

$$\Gamma_r^2+\Gamma_i^2-\frac{2r\Gamma_r}{1+r}=\frac{1-r}{1+r}\quad \textbf{or} \quad \left(\Gamma_r-\frac{r}{1+r}\right)^2+\Gamma_i^2=\frac{1}{(1+r)^2}$$

Derivation of the Smith Chart

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2}$$



This is the equation of a circle centered at

$\left(\frac{r}{1+r}, 0\right)$ and of radius $\frac{1}{1+r}$

Equating the imaginary parts gives

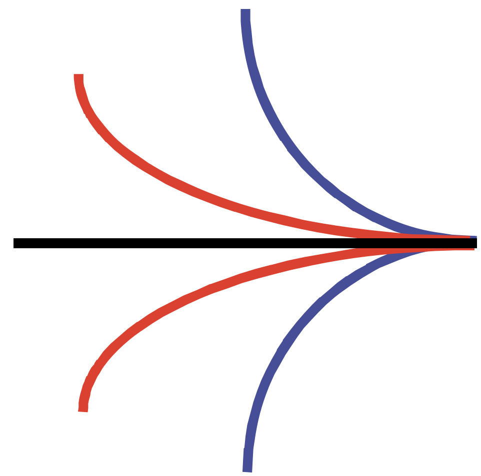
$$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$x\left[1 + \Gamma_r^2 - 2\Gamma_r + \Gamma_i^2\right] = 2\Gamma_i$ or $\Gamma_r^2 x - 2x\Gamma_r + x\Gamma_i^2 - 2\Gamma_i = -x$

Derivation of the Smith Chart

$$\Gamma_r^2 - 2\Gamma_r + 1 + \Gamma_i^2 - \frac{2\Gamma_i}{x} + \frac{1}{x^2} = \frac{1}{x^2} - 1 + 1$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$



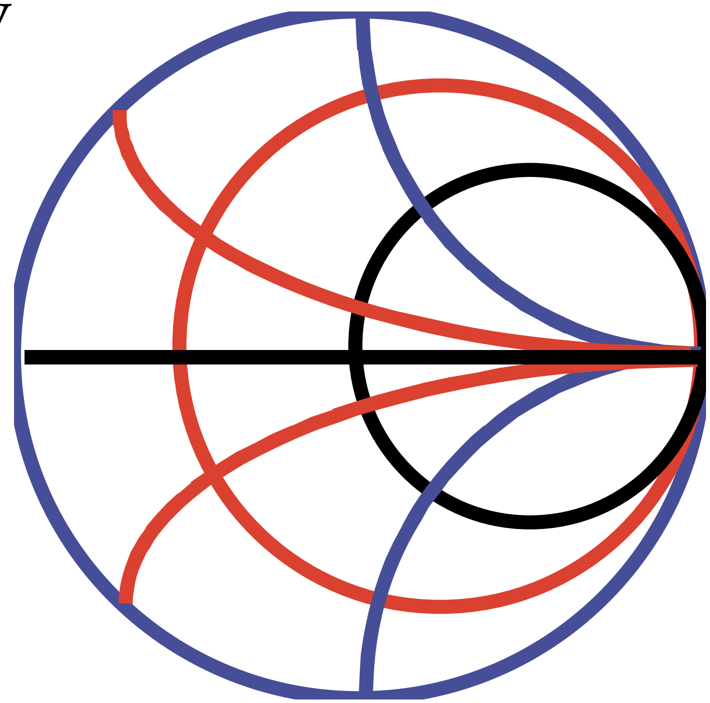
This is the equation of a circle centered at

$\left(1, \frac{1}{x}\right)$ of radius $\frac{1}{x}$

The Smith Chart

The reflection coefficient is given by

$$\Gamma = \frac{Z_n - 1}{Z_n + 1} = \frac{r - 1 + jx}{r + 1 + jx}$$



We also have

$$|\Gamma| = \left[\frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2} \leq 1$$

$$Z_n = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

$$y = \frac{1}{Z_n} = \frac{1 - \Gamma(z)}{1 + \Gamma(z)}$$

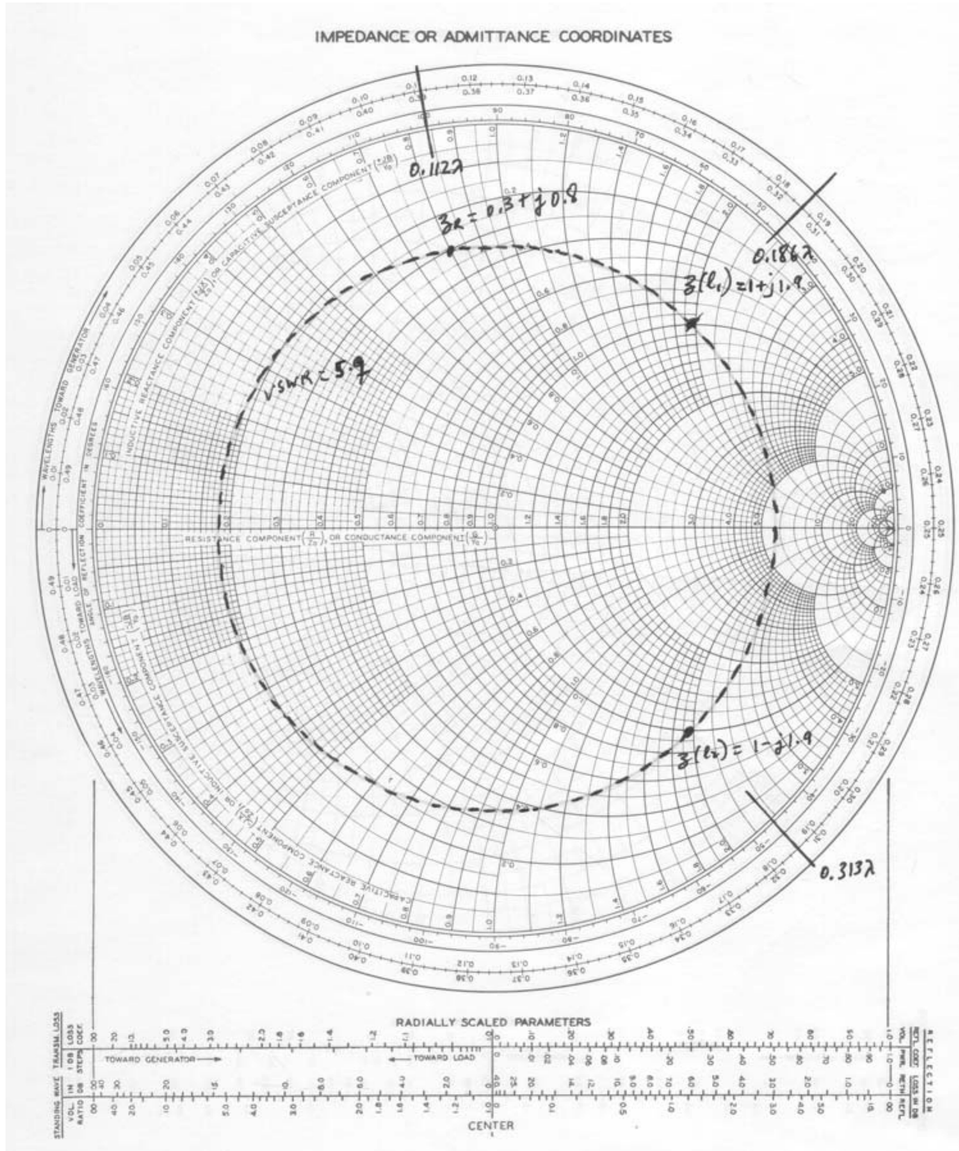
Thus, going from normalized impedance to normalized admittance corresponds to a 180 degree shift.

The Smith Chart

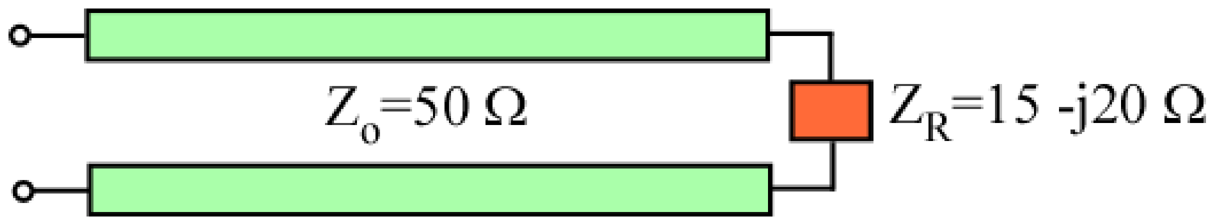
3 ways to move on the Smith chart

- Constant SWR circle → displacement along TL
- Constant resistance (conductance) circle→ addition of reactance (susceptance)
- Constant reactance (susceptance) arc → addition of resistance (conductance)

The Smith Chart



Smith Chart Example



FIND:

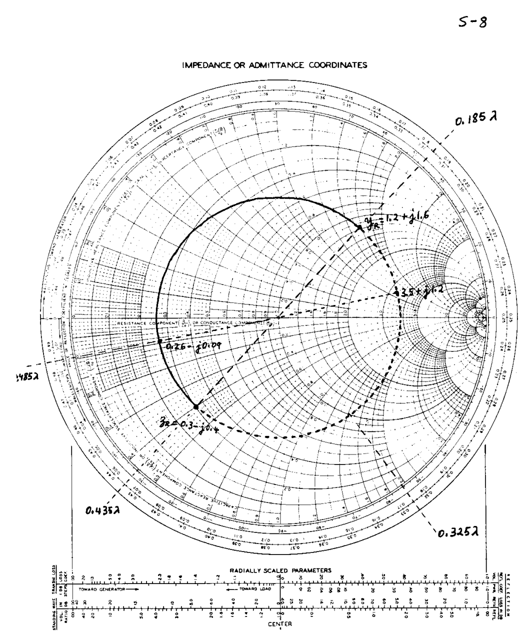
1. Reflection coefficient at load

$$z_R = 0.3 - j0.4 \Rightarrow \Gamma_R = 0.6e^{j227°}$$

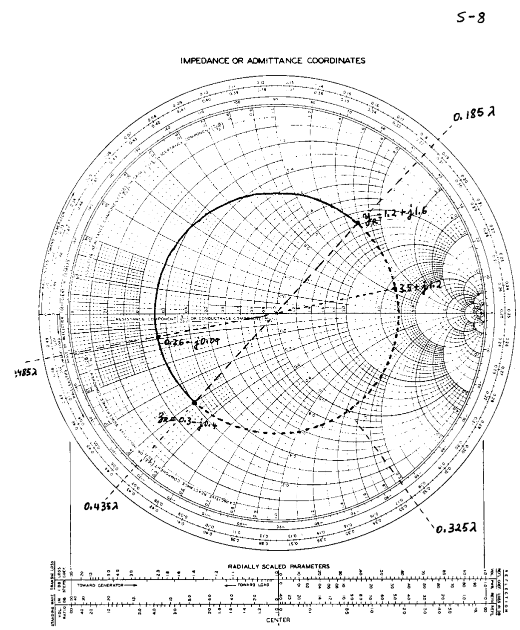
2. SWR on the line
   SWR=4.0

3. d_{min}

$$d_{min} = (0.5 - 0.435)\lambda = 0.065\lambda$$



Smith Chart Example



4. Line impedance at 0.05λ to the left

$$50(0.26 - j0.09) = 13 - j4.5\Omega$$

5. Line admittance at 0.05l

$$(3.5 + j1.2)/50 = 0.068 + j0.025\ \mho$$

6. Location nearest to load where Real[y]=1

$$0.14\lambda = 0.325\lambda - j0.185\lambda = 0.14\lambda$$

Smith Chart Example[8]