ECE 451
Advanced Microwave Measurements

Transmission Lines

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Maxwell's Equations

$$\nabla \times E = -\frac{\partial B}{\partial t}$$ **Faraday's Law of Induction**

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$ **Ampère's Law**

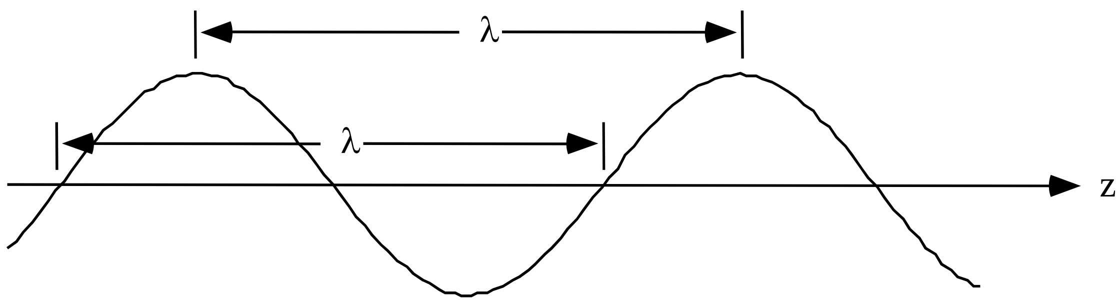
$$\nabla \cdot D = \rho$$ **Gauss' Law for electric field**

$$\nabla \cdot B = 0$$ **Gauss' Law for magnetic field**

Constitutive Relations

$$B = \mu H \qquad D = \varepsilon E$$

Why Transmission Line?



Wavelength : λ

$$\lambda = \frac{\text{propagation velocity}}{\text{frequency}}$$

Why Transmission Line?

In Free Space

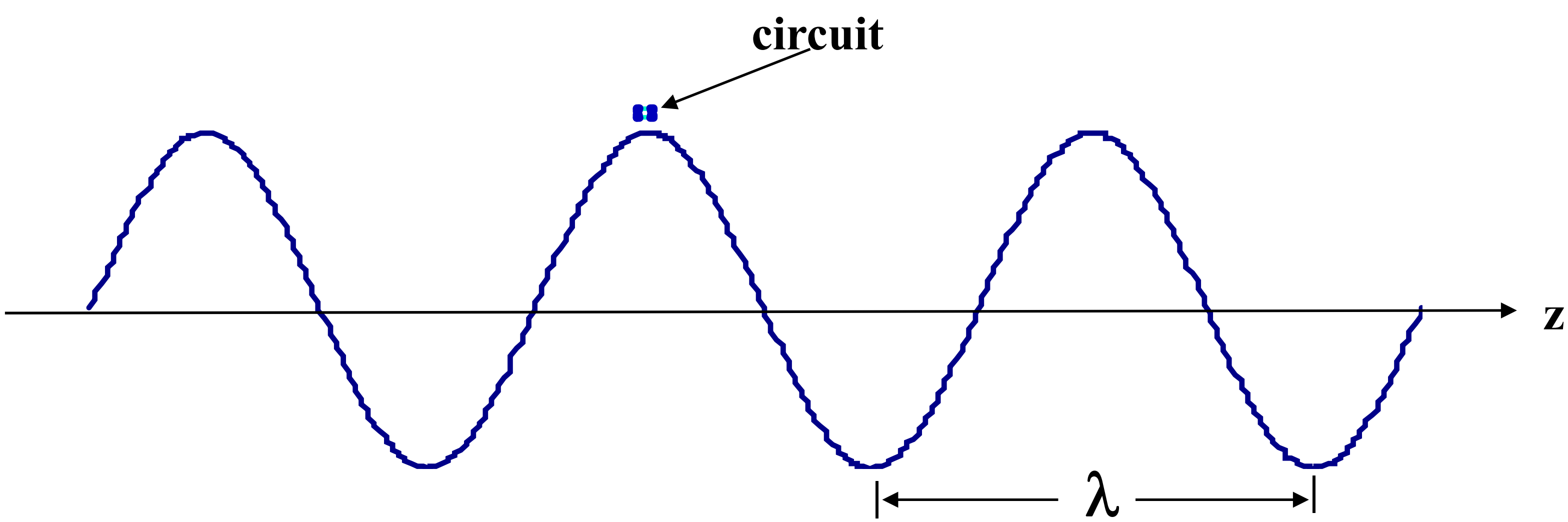
At 10 KHz : $\lambda = 30$ km

At 10 GHz : $\lambda = 3$ cm

Transmission line behavior is prevalent when the structural dimensions of the circuits are comparable to the wavelength.

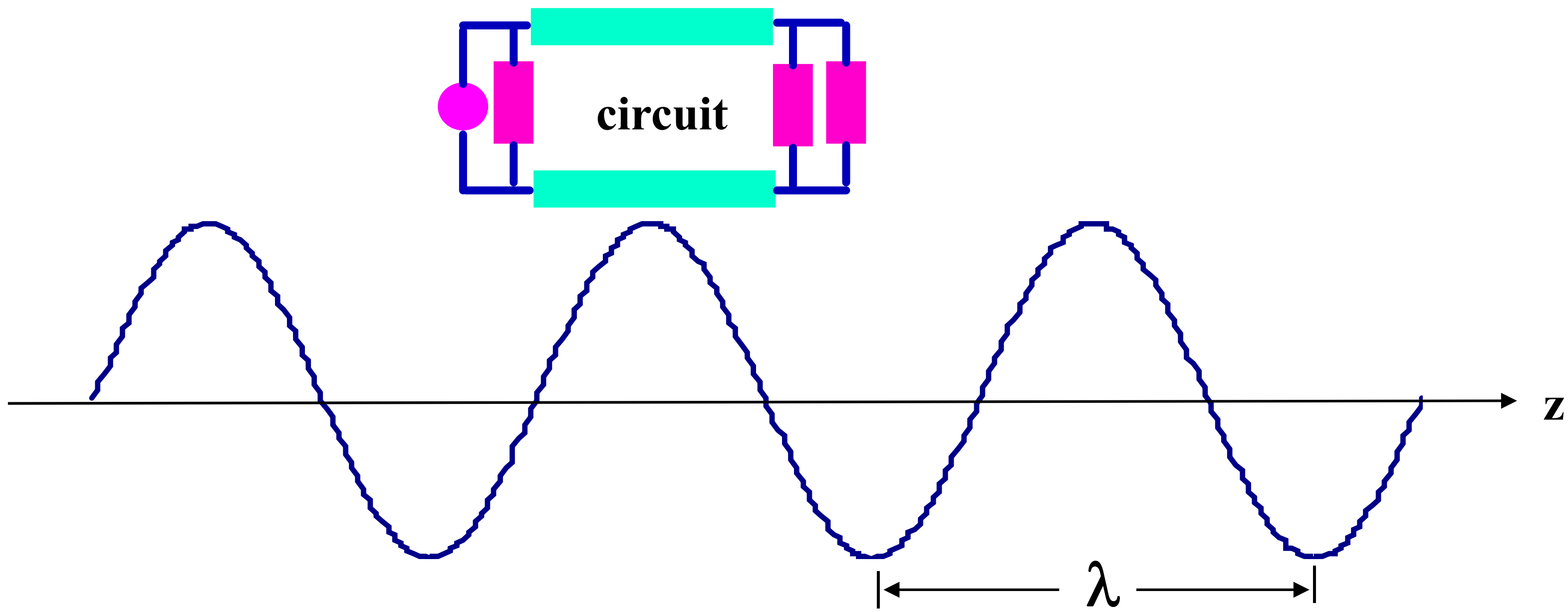
Justification for Transmission Line

Let d be the largest dimension of a circuit



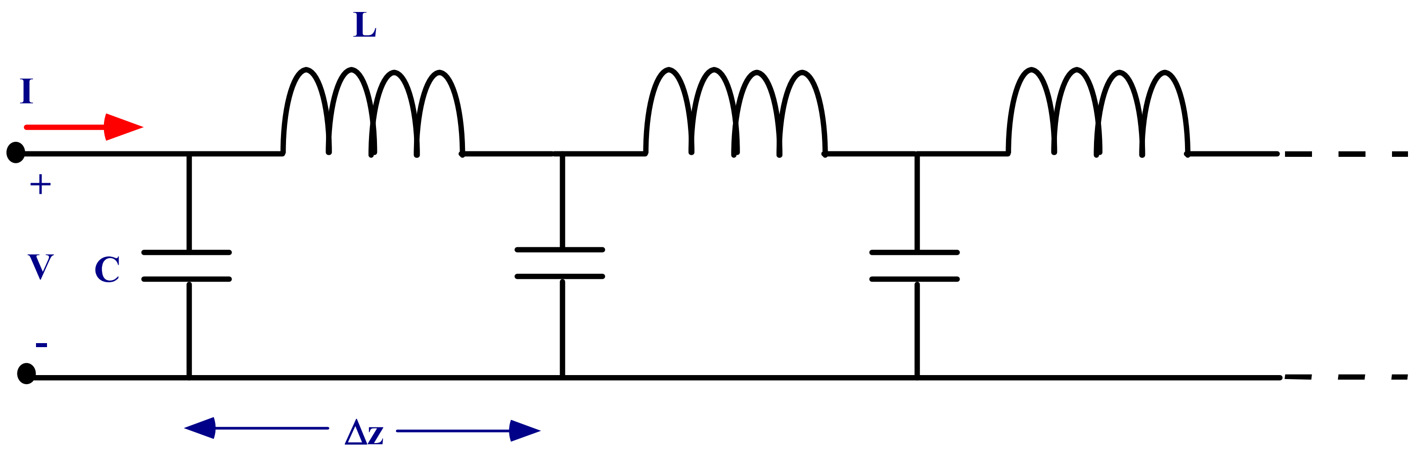
If d << λ, a lumped model for the circuit can be used

Justification for Transmission Line



If d ≈ λ, or d > λ then use transmission line model

Telegraphers' Equations



***L*: Inductance per unit length.**

***C*: Capacitance per unit length.**

$$-\frac{\partial V}{\partial z} = L\frac{\partial I}{\partial t}$$

$$-\frac{\partial I}{\partial z} = C\frac{\partial V}{\partial t}$$

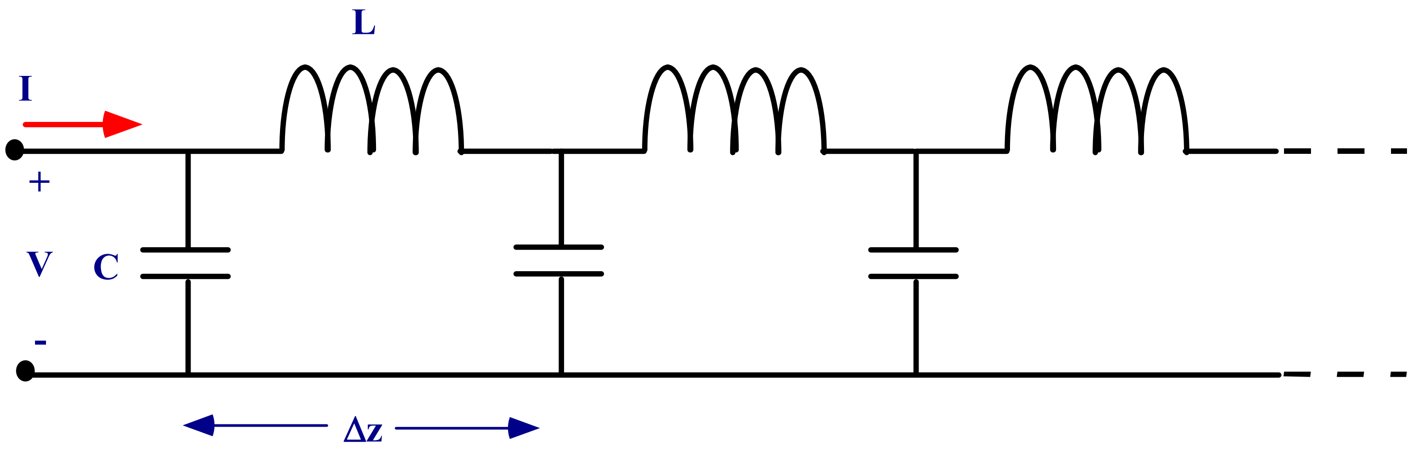
Assume time-harmonic dependence

$$V,\ I \sim e^{j\omega t}$$

$$-\frac{\partial V}{\partial z} = j\omega L I$$

$$-\frac{\partial I}{\partial z} = j\omega C V$$

TL Solutions

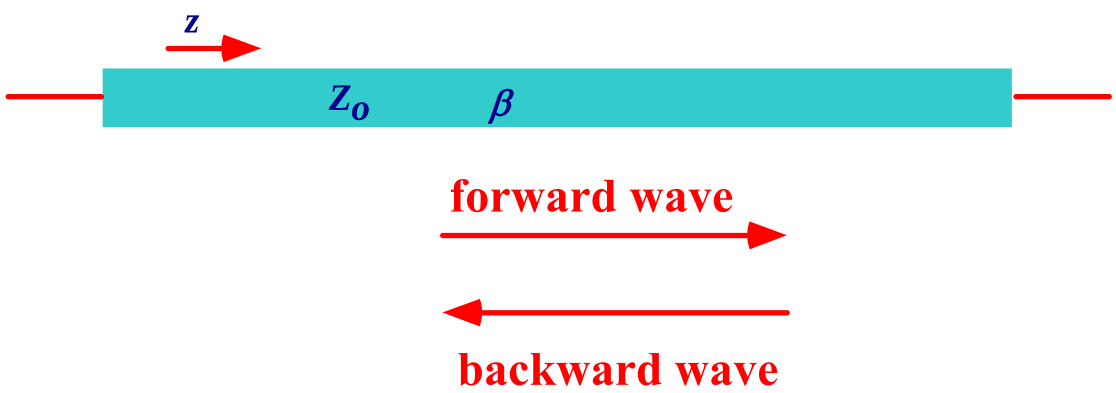


$$-\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) = -\frac{\partial^2 V}{\partial z} = j\omega L\frac{\partial I}{\partial z} = -j\omega L j\omega C V \quad \Rightarrow \quad \frac{\partial^2 V}{\partial z^2} = -\omega^2 LCV$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial I}{\partial z}\right) = -\frac{\partial^2 I}{\partial z} = j\omega C\frac{\partial V}{\partial z} = -j\omega L j\omega C I \quad \Rightarrow \quad \frac{\partial^2 I}{\partial z^2} = -\omega^2 CLI$$

TL Solutions

(Frequency Domain)



$$\beta = \omega\sqrt{LC}$$

$$V(z) = \overbrace{V_+ e^{-j\beta z}}^{Forward\ Wave} + \overbrace{V_- e^{+j\beta z}}^{Backward\ Wave}$$

$$Z_o = \sqrt{\frac{L}{C}}$$

$$I(z) = \underbrace{\frac{V_+}{Z_o} e^{-j\beta z}}_{Forward\ Wave} - \underbrace{\frac{V_-}{Z_o} e^{+j\beta z}}_{Backward\ Wave}$$

TL Solutions

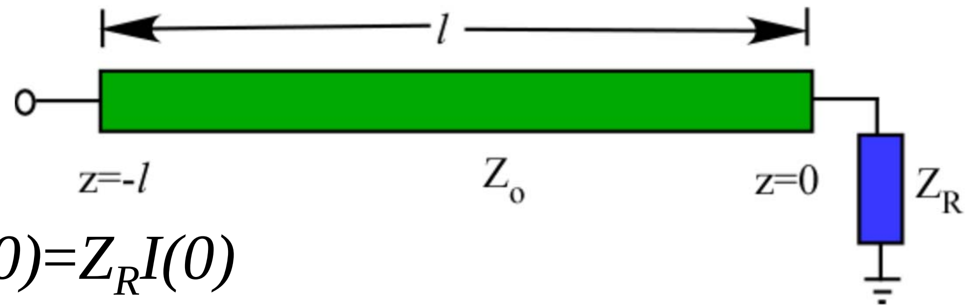
Propagation constant $\beta = \omega\sqrt{LC}$ **Propagation velocity** $v = \frac{1}{\sqrt{LC}}$

Characteristic impedance $Z_o = \sqrt{\frac{L}{C}}$ **Wavelength** $\lambda = \frac{v}{f}$

$$V(z,t) = \overbrace{V_+ \cos(\omega t - \beta z)}^{Forward\ Wave} + \overbrace{V_- \cos(\omega t + \beta z)}^{Backward\ Wave}$$

$$I(z,t) = \underbrace{\frac{V_+}{Z_o}\cos(\omega t - \beta z)}_{Forward\ Wave} - \underbrace{\frac{V_-}{Z_o}\cos(\omega t + \beta z)}_{Backward\ Wave}$$

Reflection Coefficient



At $z=0$, we have $V(0)=Z_R I(0)$

But from the TL equations:

$$V(0) = V_+ + V_-$$

$$I(0) = \frac{V_+}{Z_o} - \frac{V_-}{Z_o}$$

$$\frac{Z_R}{Z_o}\left(V_+ - V_-\right) = V_+ + V_-$$

Which gives $V_- = \Gamma_R V_+$

where $\Gamma_R = \dfrac{Z_R - Z_o}{Z_R + Z_o}$ is the load reflection coefficient

Reflection Coefficient

- If $Z_R = Z_o$, Γ_R=0, no reflection, the line is matched

- If $Z_R = 0$, short circuit at the load, Γ_R=-1

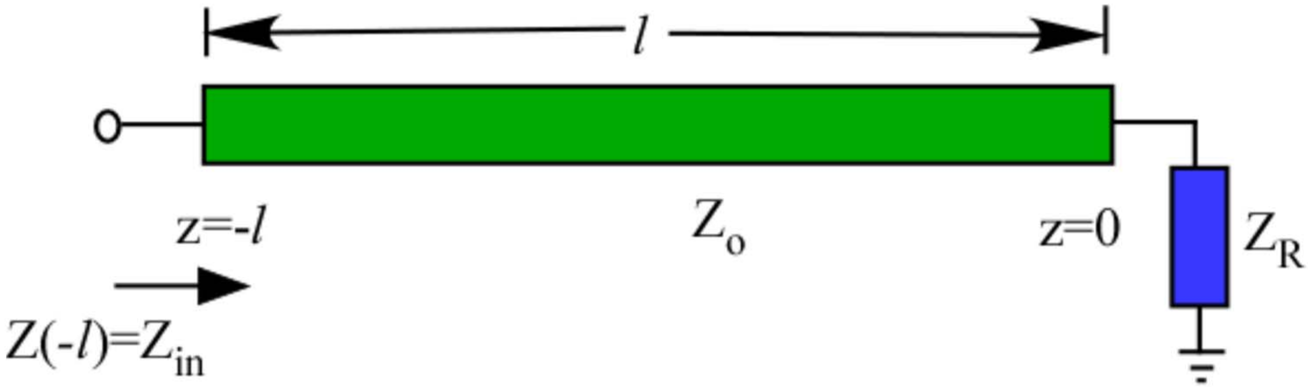
- If $Z_R \rightarrow$ inf, open circuit at the load, Γ_R=+1

V and *I* can be written in terms of Γ_R

$$V(z) = V_+\left[e^{-j\beta z} + \Gamma_R e^{+j\beta z}\right] \qquad V(z) = V_+ e^{-j\beta z}\left[1 + \Gamma_R e^{+2j\beta z}\right]$$

$$I(z) = \frac{V_+}{Z_o}\left[e^{-j\beta z} - \Gamma_R e^{+j\beta z}\right] \qquad I(z) = \frac{V_+ e^{-j\beta z}}{Z_o}\left[1 - \Gamma_R e^{+2j\beta z}\right]$$

Generalized Impedance

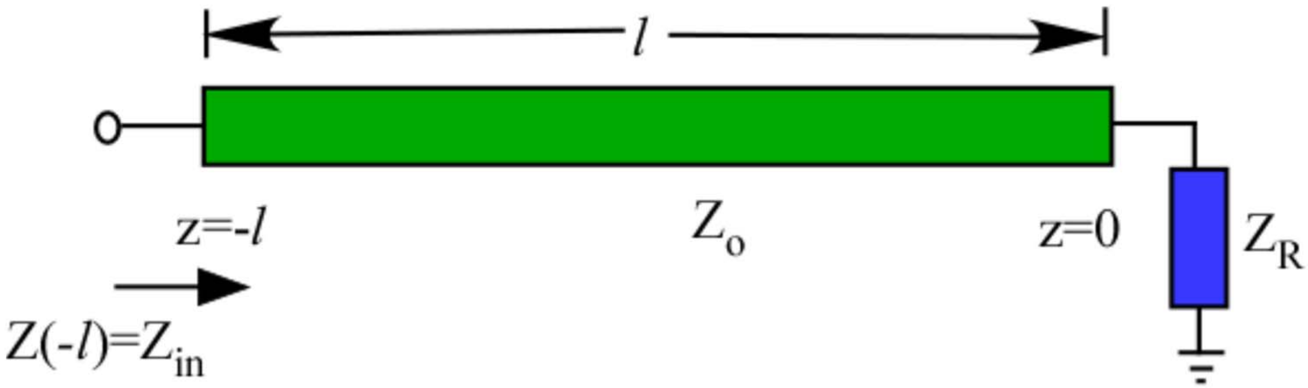


$$Z(z) = \frac{V(z)}{I(z)} = Z_o \left[\frac{e^{-j\beta z} + \Gamma_R e^{+j\beta z}}{e^{-j\beta z} - \Gamma_R e^{+j\beta z}} \right]$$

$$Z(-l) = Z_o \left[\frac{Z_R + jZ_o \tan \beta l}{Z_o + jZ_R \tan \beta l} \right]$$

Impedance transformation equation

Generalized Impedance



- **Short circuit Z_R=0, line appears inductive for 0 < *l* < λ/2**

$$Z(-l) = jZ_o \tan \beta l$$

- **Open circuit Z_R ➔ inf, line appears capacitive for 0 < *l* < λ/2**

$$Z(-l) = \frac{Z_o}{j \tan \beta l}$$

- **If *l* = λ/4, the line is a quarter-wave transformer**

$$Z(-l) = \frac{Z_o^2}{Z_R}$$

Generalized Reflection Coefficient

$$\Gamma(z) = \frac{\text{Backward traveling wave at } z}{\text{Forward traveling wave at } z} = \frac{V_b(z)}{V_f(z)}$$

$$\Gamma(z) = \frac{V_- e^{+j\beta z}}{V_+ e^{-j\beta z}} = \frac{V_-}{V_+} e^{+2j\beta z} = \Gamma_R e^{+2j\beta z}$$

Reflection coefficient transformation equation ⇒ $\Gamma(-l) = \Gamma_R e^{-2j\beta l}$

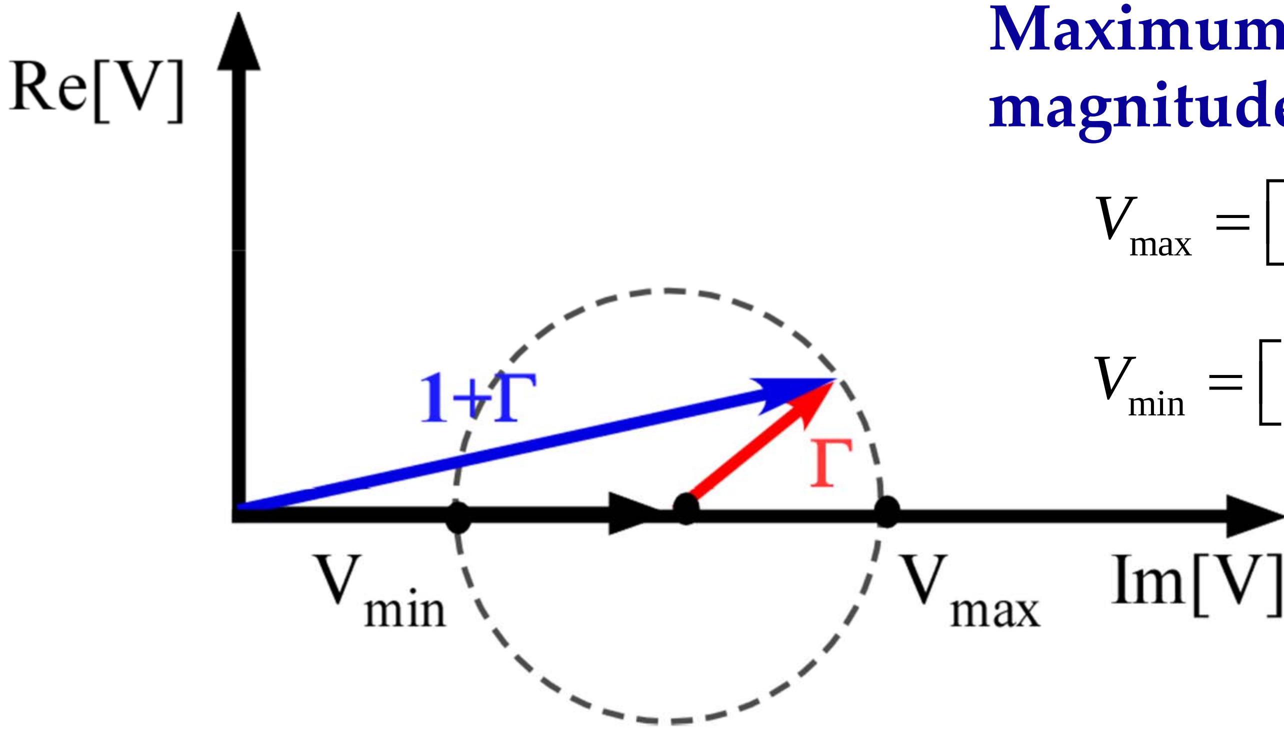
$$Z(z) = Z_o \frac{1+\Gamma(z)}{1-\Gamma(z)} \qquad \Gamma(z) = \frac{Z(z) - Z_o}{Z(z) + Z_o}$$

Voltage Standing Wave Ratio (VSWR)

$$V(z) = V_+ e^{-j\beta z}\left[1+\Gamma_R e^{+2j\beta z}\right]$$

We follow the magnitude of the voltage along the TL

$$|V(z)| = \left|V_+ e^{-j\beta z}\right|\left|1+\Gamma_R e^{+2j\beta z}\right| = |V_+|\left|1+\Gamma_R e^{+2j\beta z}\right|$$



Maximum and minimum magnitudes given by

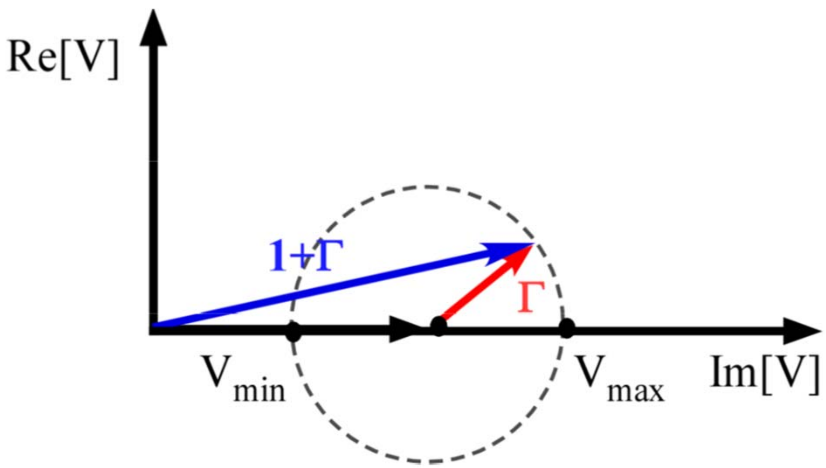
$$V_{\max} = \left[1+|\Gamma_R|\right]$$

$$V_{\min} = \left[1-|\Gamma_R|\right]$$

Voltage Standing Wave Ratio (VSWR)

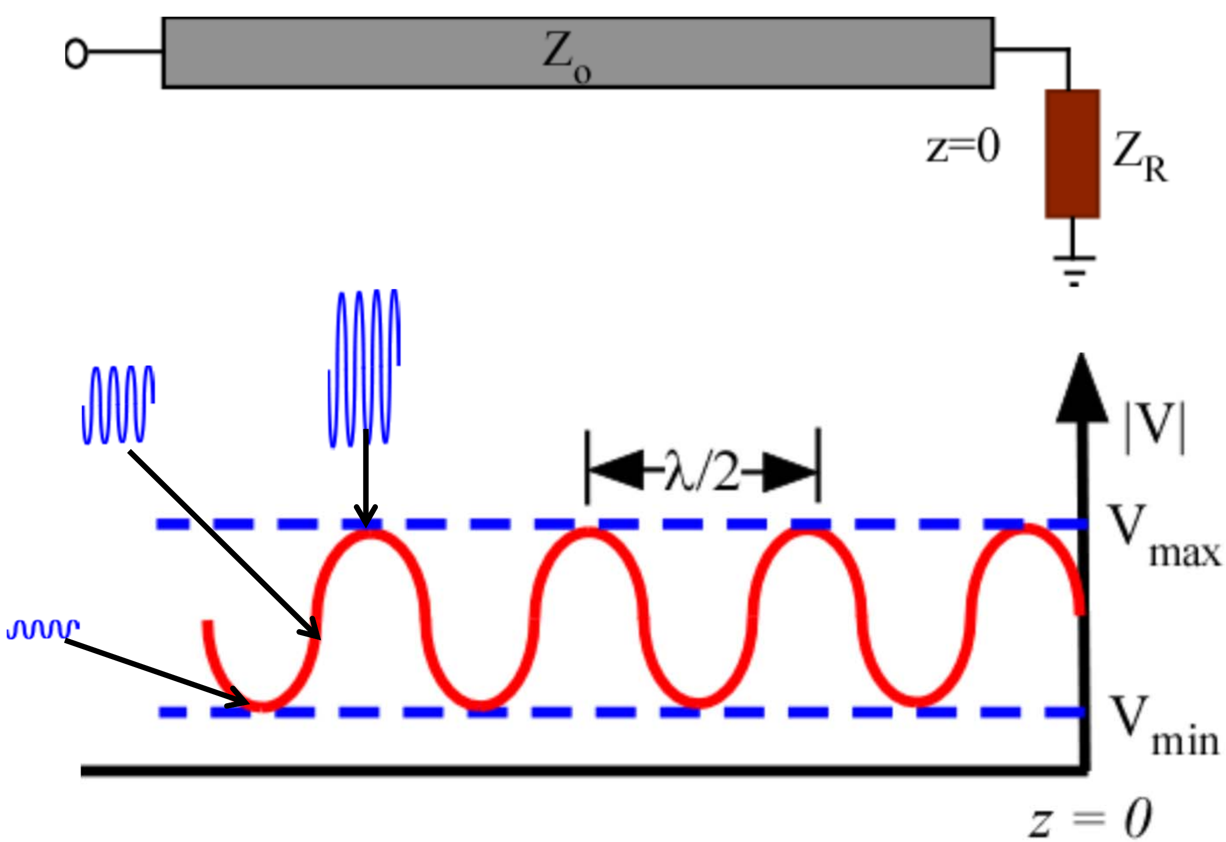
Define Voltage Standing Wave Ratio as:

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1+\left|\Gamma_R\right|}{1-\left|\Gamma_R\right|}$$



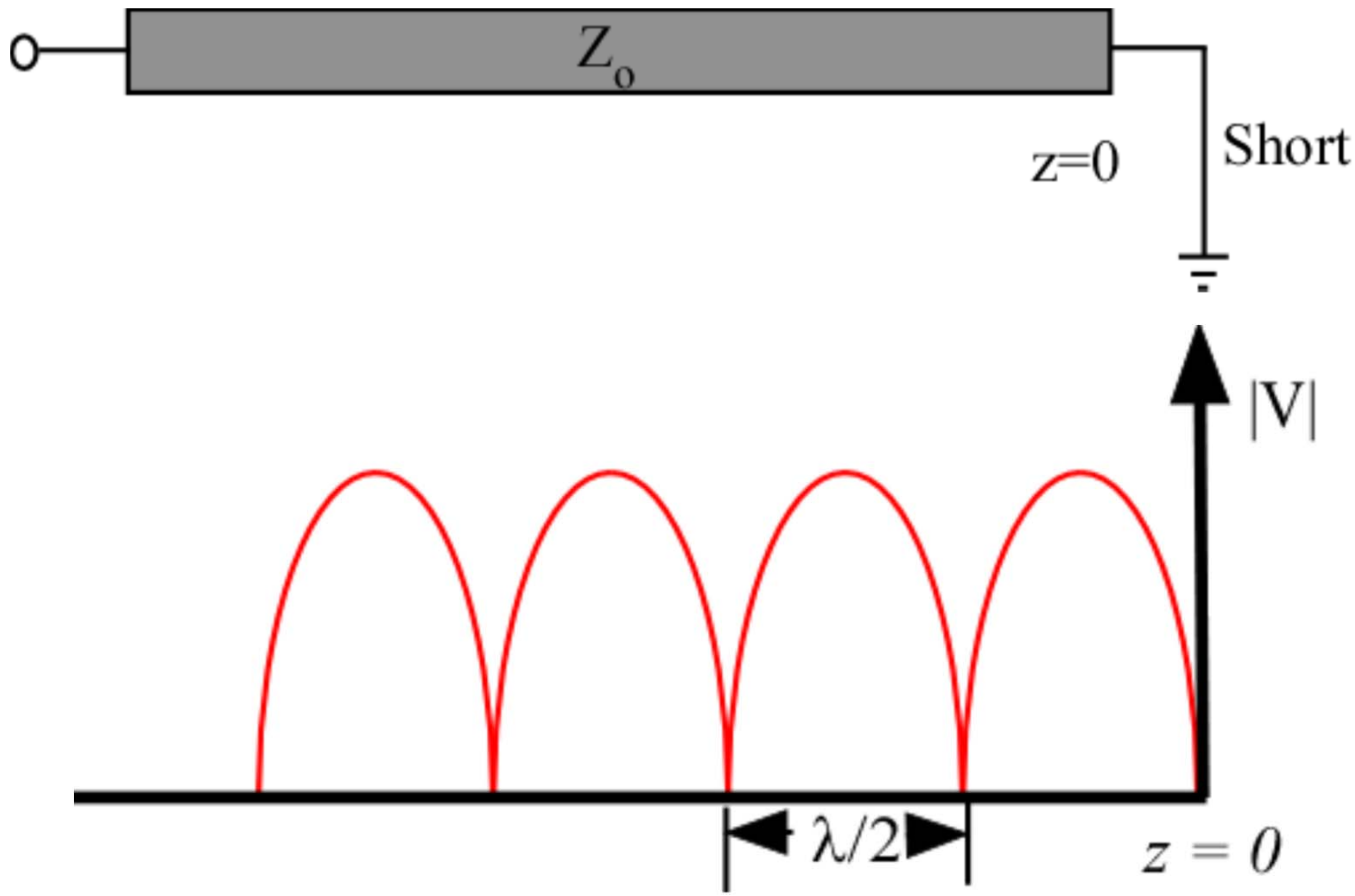
It is a measure of the interaction between forward and backward waves

VSWR – Arbitrary Load



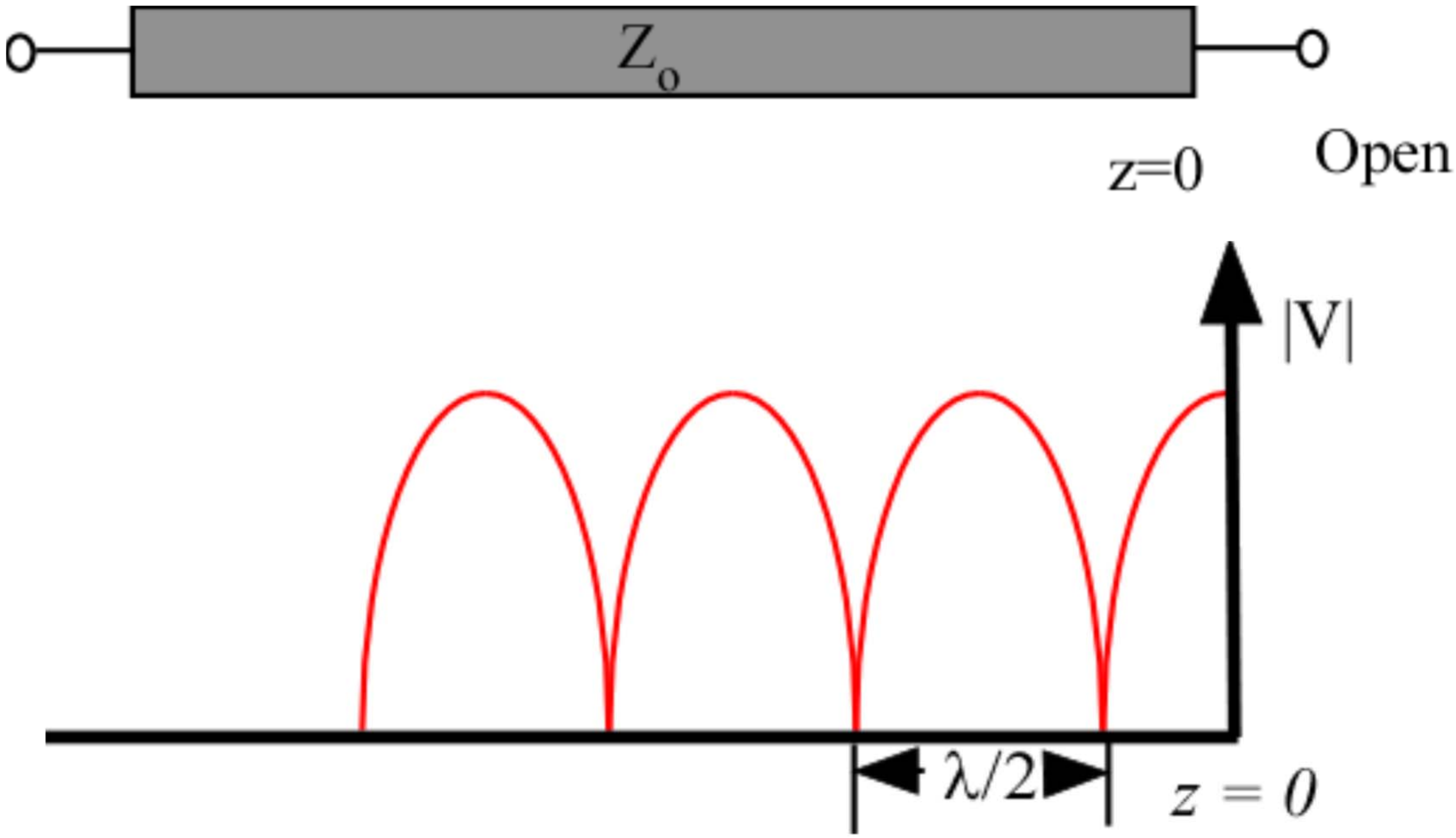
Shows variation of amplitude along line

VSWR – For Short Circuit Load



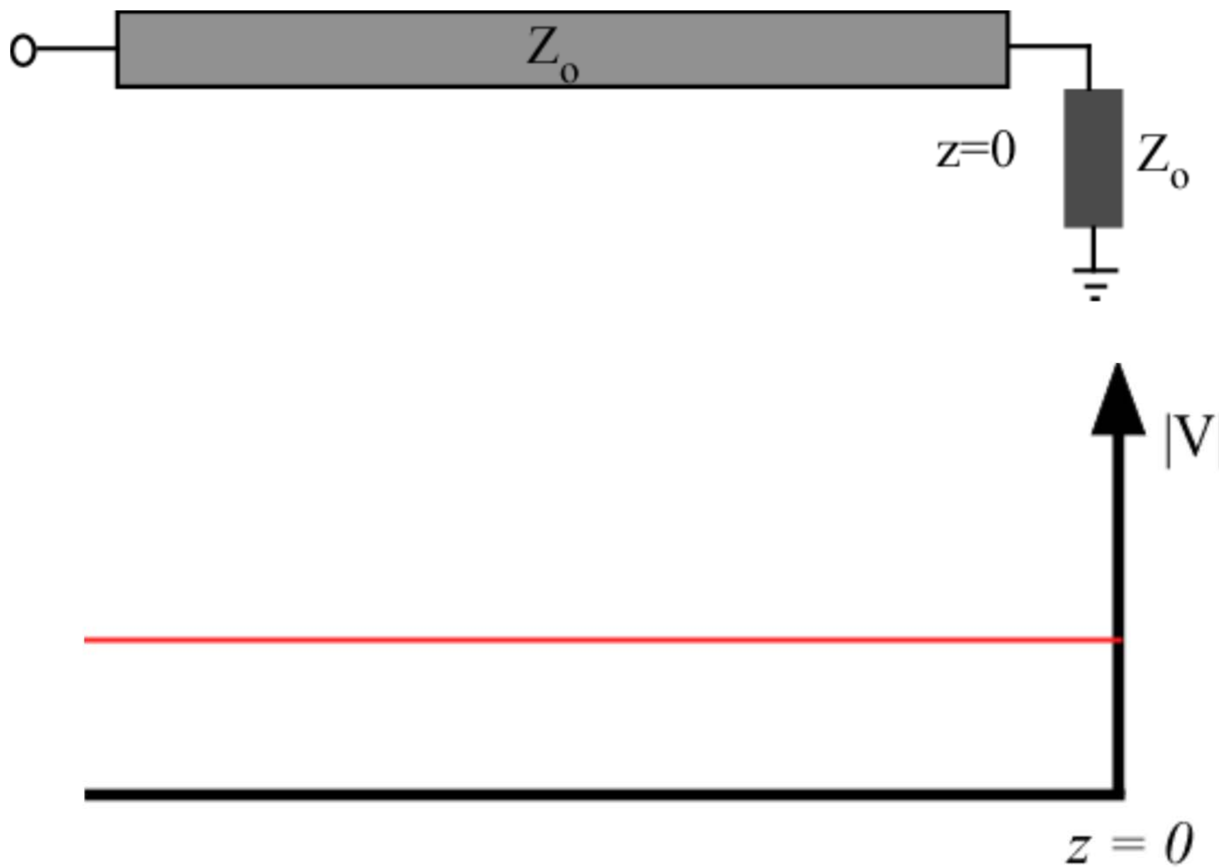
Voltage minimum is reached at load

VSWR – For Open Circuit Load



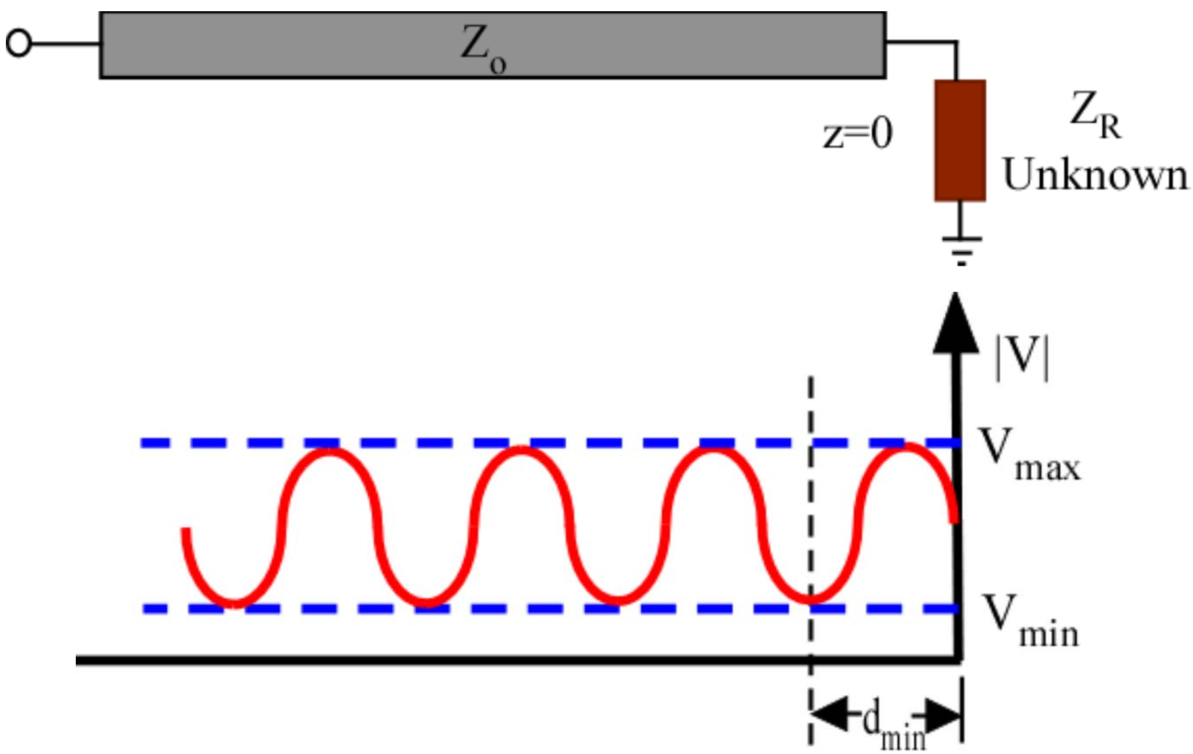
Voltage maximum is reached at load

VSWR – For Open Matched Load



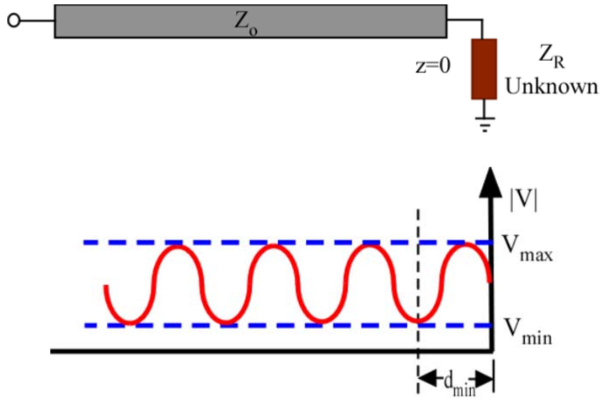
No variation in amplitude along line

Application: Slotted-Line Measurement



- Measure VSWR = V_{max}/V_{min}
- Measure location of first minimum

Application: Slotted-Line Measurement



At minimum, $\Gamma(z) = \text{pure real} = -|\Gamma_R|$

Therefore, $\Gamma(-d_{\min}) = \Gamma_R e^{-2j\beta d_{\min}} = -|\Gamma_R|$

So, $\Gamma_R = -|\Gamma_R| e^{+2j\beta d_{\min}}$

Since $|\Gamma_R| = \dfrac{VSWR-1}{VSWR+1}$ **then** $\Gamma_R = -\left(\dfrac{VSWR-1}{VSWR+1}\right) e^{+2j\beta d_{\min}}$

and $Z_R = Z_o \left(\dfrac{1+\Gamma_R}{1-\Gamma_R}\right)$

Summary of TL Equations

Voltage

$$V(z) = V_+ e^{-j\beta z}\left[1 + \Gamma_R e^{+2j\beta z}\right]$$

Current

$$I(z) = \frac{V_+}{Z_o} e^{-j\beta z}\left[1 - \Gamma_R e^{+2j\beta z}\right]$$

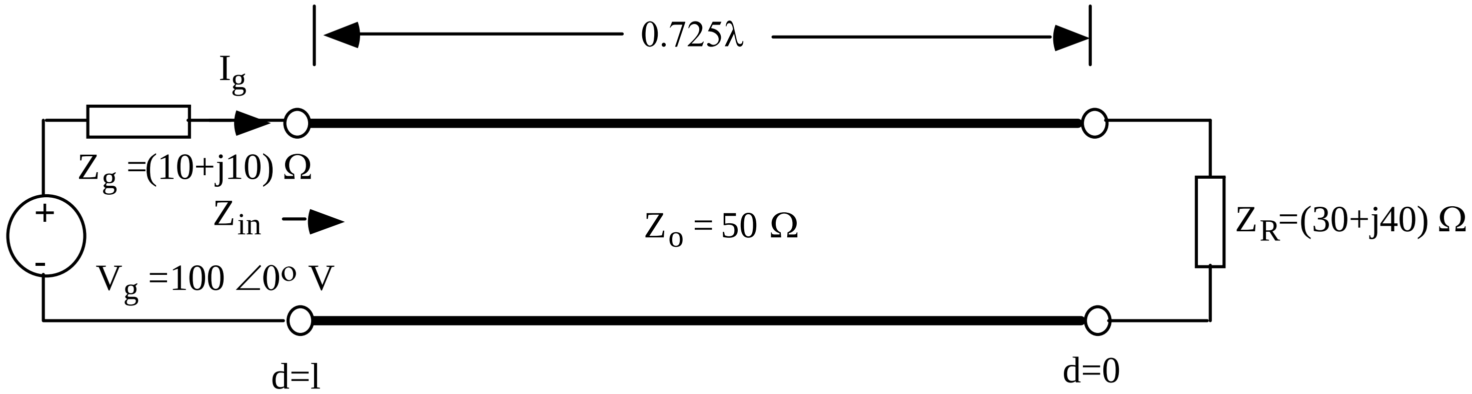
Impedance Transformation ➔ $Z(-l) = Z_o\left[\frac{Z_R + jZ_o \tan\beta l}{Z_o + jZ_R \tan\beta l}\right]$

Reflection Coefficient Transformation ➔ $\Gamma(-l) = \Gamma_R e^{-2j\beta l}$

Reflection Coefficient – to Impedance ➔ $Z(z) = Z_o \frac{1+\Gamma(z)}{1-\Gamma(z)}$

Impedance to Reflection Coefficient ➔ $\Gamma(z) = \frac{Z(z) - Z_o}{Z(z) + Z_o}$

Example 1



Consider the transmission line system shown in the figure above.
(a) Find the input impedance Z_{in}.

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o} = \frac{(30 + j40) - 50}{(30 + j40) + 50} = 0.5\angle 90^\circ$$

$$\Gamma(d = l) = \Gamma_R e^{-2j\beta l} = 0.5\angle 90^\circ e^{-j\frac{4\pi}{\lambda}.0.725\lambda} = 0.5\angle -72^\circ$$

Example 1 – Cont'

$$Z_{in} = Z_o\left[\frac{1+\Gamma(l)}{1-\Gamma(l)}\right] = 50\left[\frac{1+0.5\angle-72^\circ}{1-0.5\angle-72^\circ}\right]\Omega$$

$$Z_{in} = 64.361\angle-51.738^\circ\,\Omega = \left(39.86 - j50.54\right)\Omega$$

(b) Find the current drawn from the generator

$$I_g = \frac{V_g}{Z_g + Z_{in}} = \frac{100\angle 0^\circ}{(10+j10)(39.86-j50.54)} = 1.552\angle 39.11^\circ\ A$$

$$I_g = 1.207 + j0.981\ A$$

Example 1 – Cont'

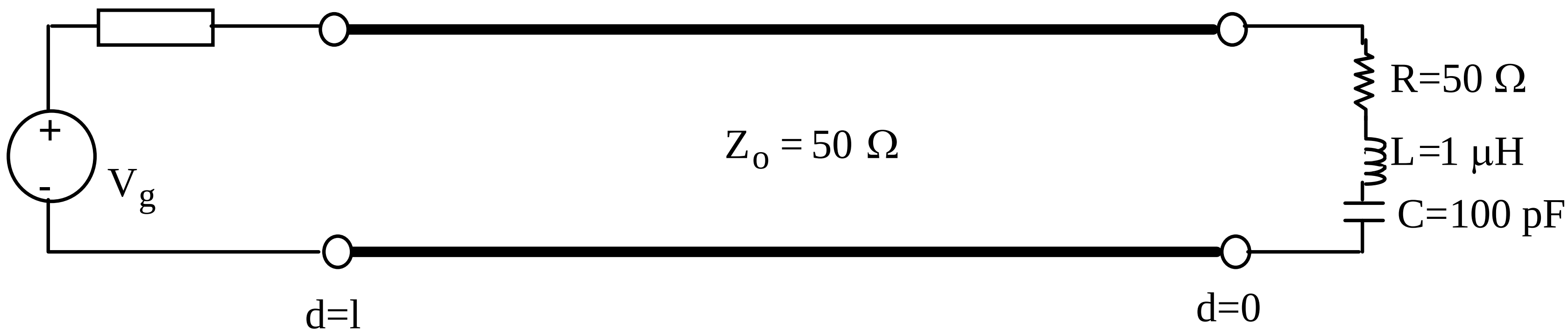
c) Find the time-average power delivered to the load

$$V(l) = Z_{in} I_g = 64.361\angle 51.73^\circ \times 1.5562\angle 39.11^\circ = 100.159\angle -12.62^\circ \frac{1}{2}$$

$$< P >= \frac{1}{2}\text{Real}\left[V(l) I_g^*\right] = \frac{1}{2}\text{Real}\left[100.159\angle -12.62^\circ \times 1.5562\angle -39.11^\circ\right]$$

$$< P >= \frac{1}{2}\text{Real}\left[V(l) I_g^*\right] = 48.26\, W$$

Example 1 – Cont'



In the system shown above, a line of characteristic impedance 50 Ω is terminated by a series R, L, C circuit having the values R = 50 Ω, L = 1 μH, and C = 100 pF.

(d) Find the source frequency f_o for which there are no standing waves on the line.

Example 1 – Cont'

No standing wave on the line if $Z_R = R = Z_o$ which occurs at the resonant frequency of the RLC circuit. Thus,

$$f_o = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10^{-6}\times 10^{-10}}} = \frac{10^8}{2\pi}\ Hz = 15.92\ MHz$$