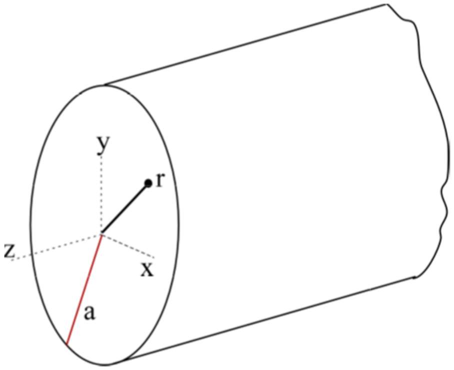
ECE 451
Advanced Microwave Measurements

Circular and Coaxial Waveguides

Jose E. Schutt-Aine
Electrical & Computer Engineering
University of Illinois
jesa@illinois.edu

Circular Waveguide - Fields

For a waveguide with arbitrary cross section, it is known that



TE Modes $$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}=\left[\beta_z^2-\omega^2\mu\varepsilon\right]H_z \quad \textbf{(1)}$$

TM Modes $$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}=\left[\beta_z^2-\omega^2\mu\varepsilon\right]E_z \quad \textbf{(2)}$$

We first assume TM modes in cylindrical coordinates:

$$\underbrace{\frac{\partial^2 E_z}{\partial r^2}+\frac{1}{r}\frac{\partial E_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2}}_{\nabla_{tr}^2 E_z}+\left(\gamma^2-\omega^2\mu\varepsilon\right)E_z=0$$

See Reference [6].

Circular Waveguide – TM Modes

Solution will be in the form

$$E_z(r,\phi) = f(r)g(\phi)$$

Which after substitution gives

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right)+h^2r^2=-\frac{1}{g}\frac{d^2g}{d\phi^2} \qquad \textbf{(3)}$$

where $h^2=\gamma^2+\omega^2\mu\varepsilon$

For equality in (3) to hold, both sides must be equal to the same constant say n^2 where n is an integer in view of the azimuthal symmetry since the fields must be periodic in ϕ.

Circular Waveguide – TM Modes

$$\frac{d^2 g}{d\phi^2} + n^2 g = 0 \quad \textbf{(4)}$$

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\left(h^2 - \frac{n^2}{r^2}\right) f = 0 \quad \textbf{(5)}$$

Solution of (4) is of the form

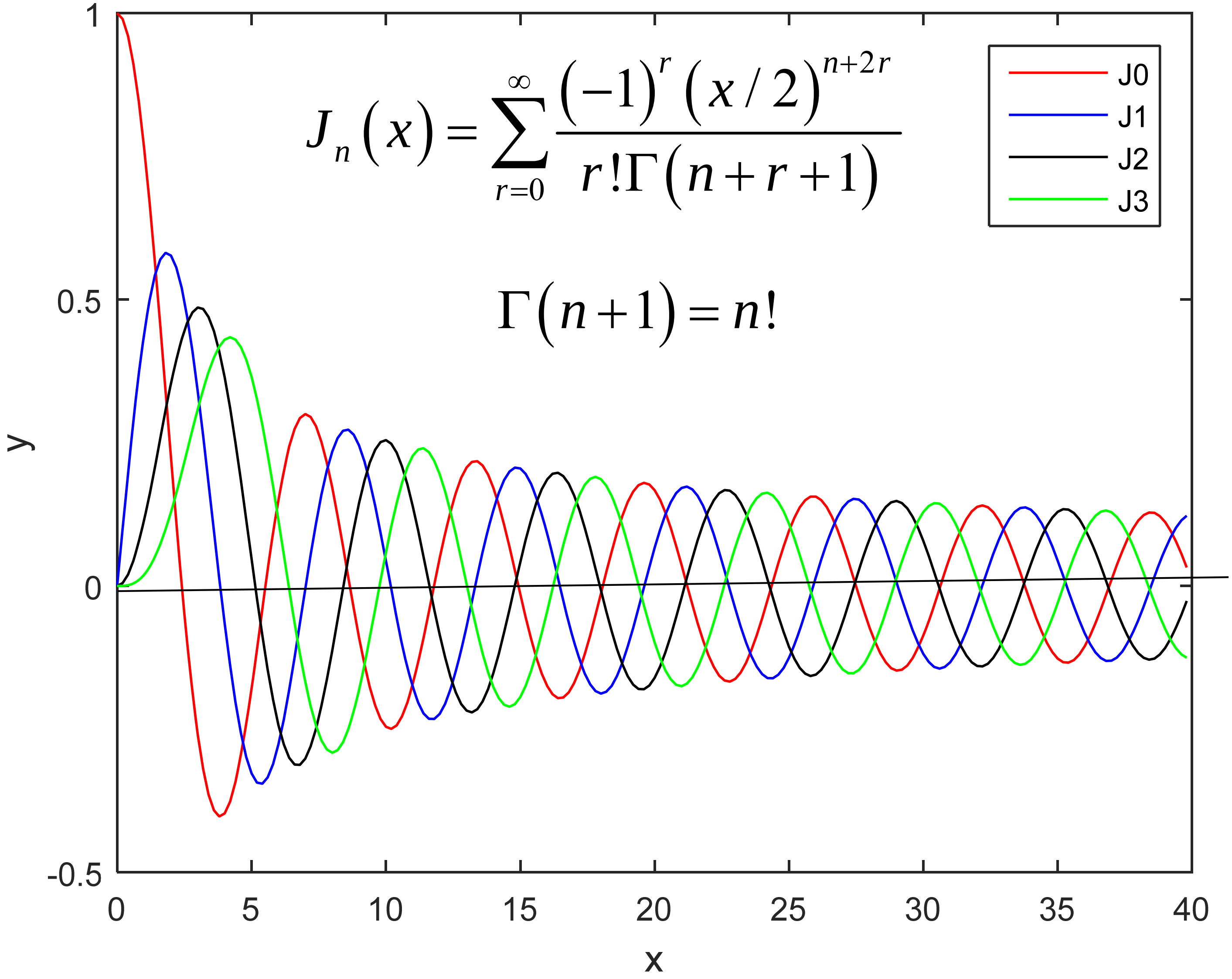
$$g(\phi) = C_1 \cos(n\phi) + C_2 \sin(n\phi) \quad \textbf{(6)}$$

(5) is Bessel's equation and has solution

$$f(r) = C_3 J_n(hr) + C_4 Y_n(hr) \quad \textbf{(7)}$$

J_n and Y_n are the n^{th} order Bessel functions of the first and second kinds respectively

Bessel Functions of the First Kind



$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\Gamma(n+r+1)}$$

$$\Gamma(n+1) = n!$$

Circular Waveguide – TM Modes

Y_n has singularity at 0 and must consequently be discarded ➔ $C_4 = 0$. The general solution then becomes

$$E_z(r,\phi) = C_3 J_n(hr)\left[C_1 \cos(n\phi) + C_2 \sin(n\phi)\right]$$

Since the origin for ϕ is arbitrary, the expression can be written as:

$$E_z(r,\phi) = C_n J_n(hr)\cos(n\phi)$$

where C_n is a constant. The boundary condition $E_{tan} = 0$ requires that

$$E_z(r,\phi) = 0 \;\; for \; r = a$$

Solution exists for only discrete values of h such that

$$J_n(ha) = 0$$

Circular Waveguide – TM Modes

ha must be a root of the n^{th} order Bessel function. If we assume that t_{nl} is the l^{th} root of J_n, we can define a set of eigenvalues h_{nl} for the TM modes so that:

$$h_{TM_{nl}} = \frac{t_{nl}}{a}$$

l^{th} root of $J_n(.)=0$

n →

l ↓

	0	1	2
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	13.323	11.620

Each choice of n and l specifies a particular solution or *mode*

n is related to the number of circumferential variations and l describes the number of radial variations of the field.

Circular Waveguide – TM Modes

The propagation constant of the nl^{th} propagating TM mode is:

$$\beta_{TM_{nl}} = \left[\omega^2 \mu\varepsilon - \left(\frac{t_{nl}}{a} \right)^2 \right]^{1/2}$$

The propagation occurs for $\lambda < \lambda_{cTMnl}$ or $f > f_{cTMnl}$ where the cutoff frequency and wavelength can be found from $\gamma = 0$ as:

$$\lambda_{cTMnl} = \frac{2\pi a}{t_{nl}} \qquad f_{cTMnl} = \frac{t_{nl}}{2\pi a\sqrt{\mu\varepsilon}}$$

The other field components can be obtained from E_z

$$E_z = C_n J_n\left(\frac{t_{nl}}{a} r \right) \cos\left(n\phi\right) e^{-j\beta_{nl} z}$$

Circular Waveguide – TE Modes

The solutions for the TE modes can be found in a similar manner except that we solve for $H_z(r,\phi)$ to get:

$$H_z\left(r,\phi\right)=C_nJ_n\left(hr\right)\cos\left(n\phi\right)$$

To apply the boundary condition $E_{tan} = 0$, we require

$$\frac{\partial H_z}{\partial r} \text{ to be 0 at } r = a$$

We must have $\hat{n}\cdot\nabla_{tr}H_z=\frac{\partial H_z}{\partial r}=0 \quad at \quad r=a$

For this, we need the zeros of $J_n'(u)$ given by s_{nl}. The propagation constant, cutoff frequency and wavelength have the same expressions as in the TM case with $t_{nl} \rightarrow s_{nl}$.

Circular Waveguide – TE Modes

The propagation constant of the *nl*th propagating TE mode is:

$$\beta_{TE_{nl}} = \left[\omega^2 \mu\varepsilon - \left(\frac{s_{nl}}{a} \right)^2 \right]^{1/2}$$

l^{th} root of $J_n'(.)=0$

n →

l ↓

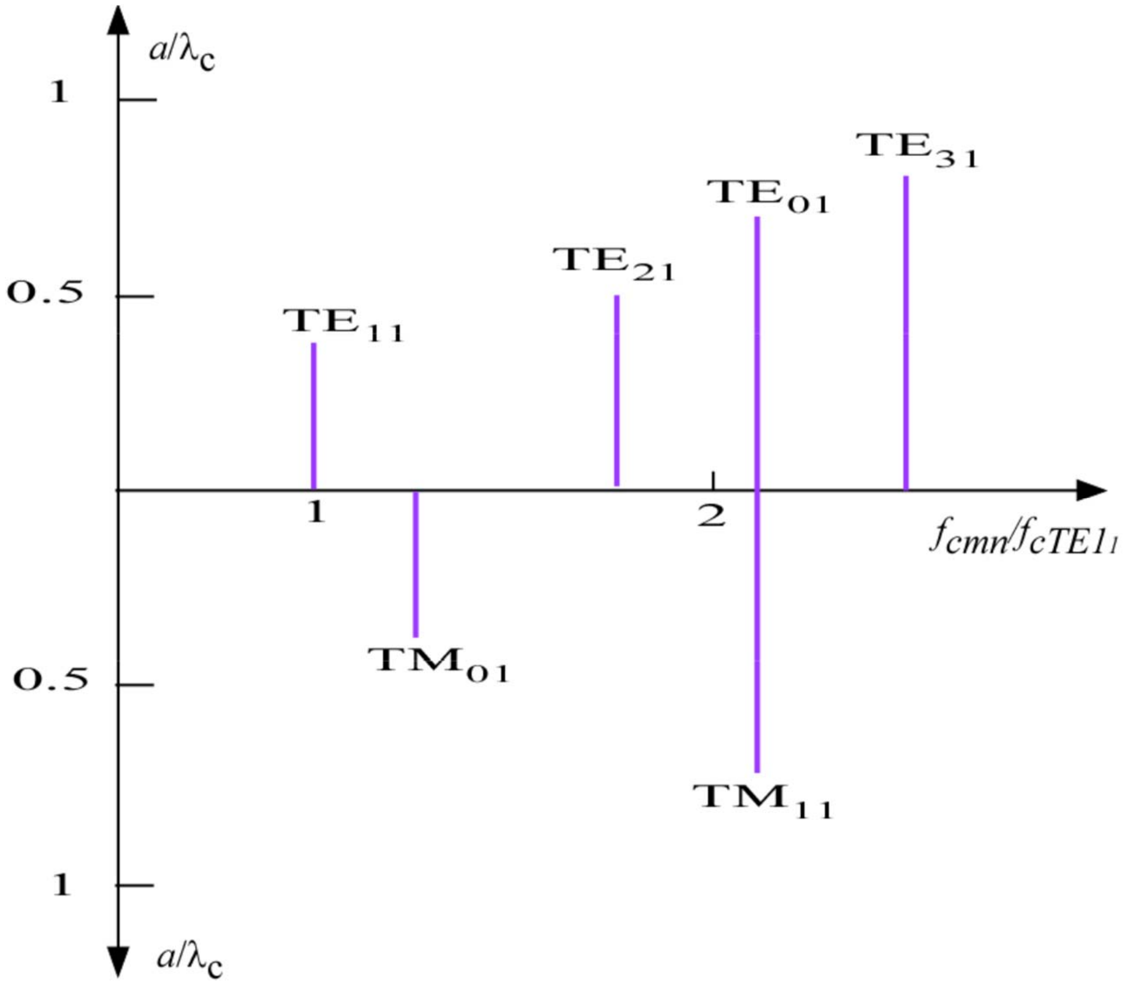
	0	1	2
1	3.832	1.841	3.054
2	7.016	5.331	6.706
3	10.173	8.536	9.969

From the tables, it can be seen that the lowest cutoff frequency is the TE_{11} mode.

and for TE modes,

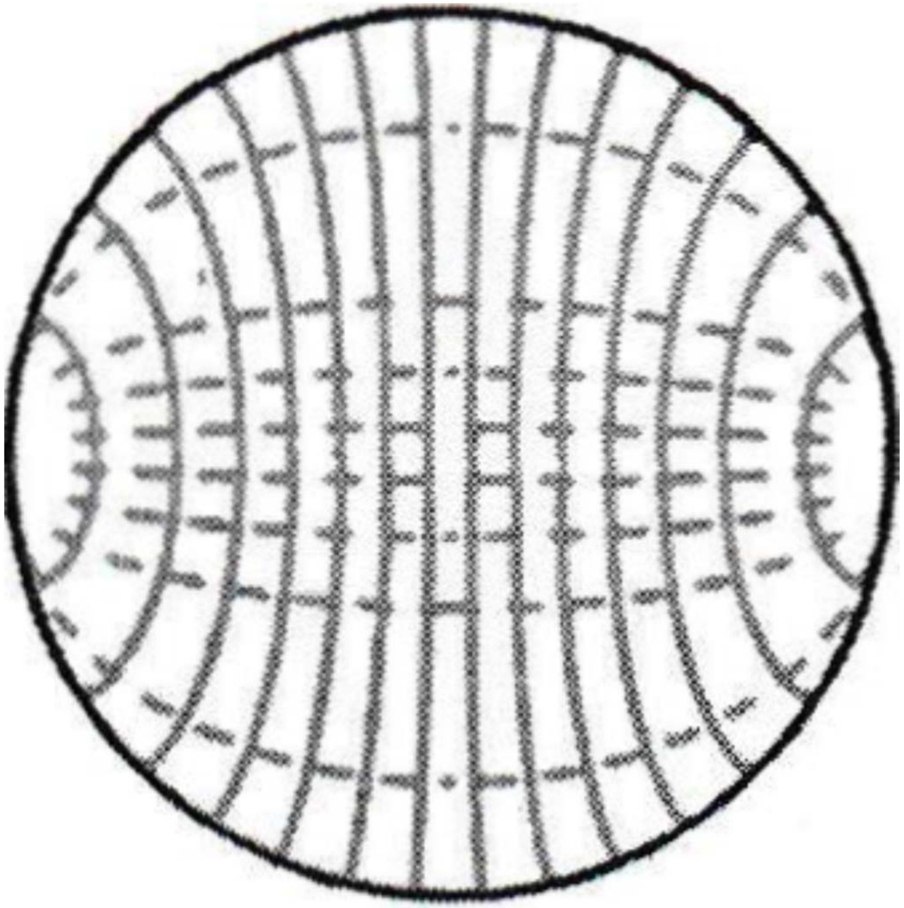
$$H_z = C_n J_n \left(\frac{s_{nl}}{a} r \right) \cos(n\phi) e^{-j\beta_{nl} z}$$

Circular Waveguide – TE & TM Modes



See Reference [6].

TE_{11} Mode in Circular Waveguide

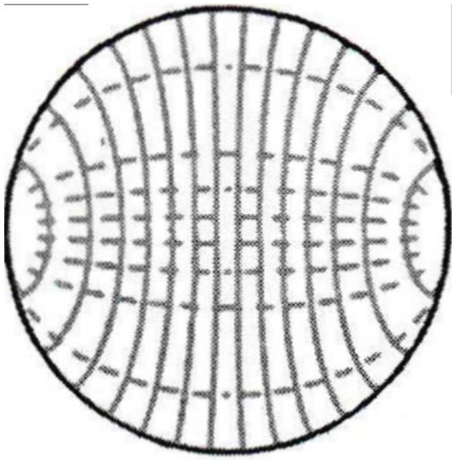


E ———

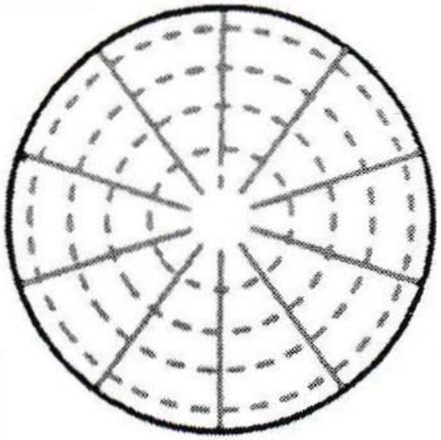
H – – – – -

See Reference [1].

Modes in Circular Waveguide



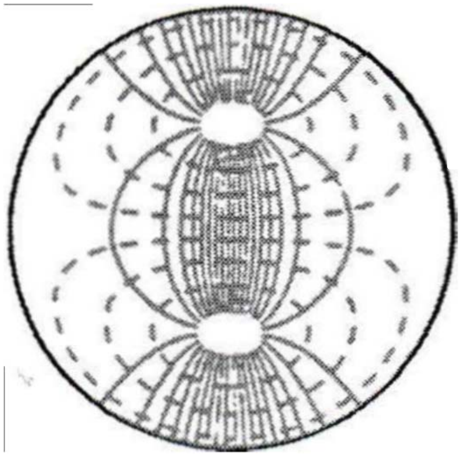
TE_{11}



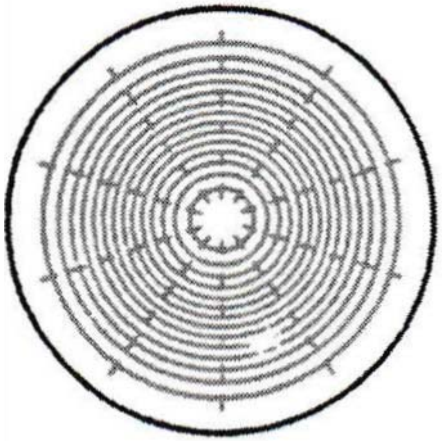
TM_{01}

E ————

H – – – – –



TM_{11}



TE_{01}

See Reference [1].

Example: Circular Waveguide Design

Design an air-filled circular waveguide such that only the dominant mode will propagate over a bandwidth of 10 GHz.

Solution: the cutoff frequency of the TE_{11} mode is the lower bound of the bandwidth.

$$f_{cTE_{11}} = \frac{1.8412c}{2\pi a}$$

The next mode is the TM_{01} with cutoff frequency:

$$f_{cTM_{01}} = \frac{2.4049c}{2\pi a}$$

Example: Circular Waveguide Design

The BW is the difference between these two frequencies

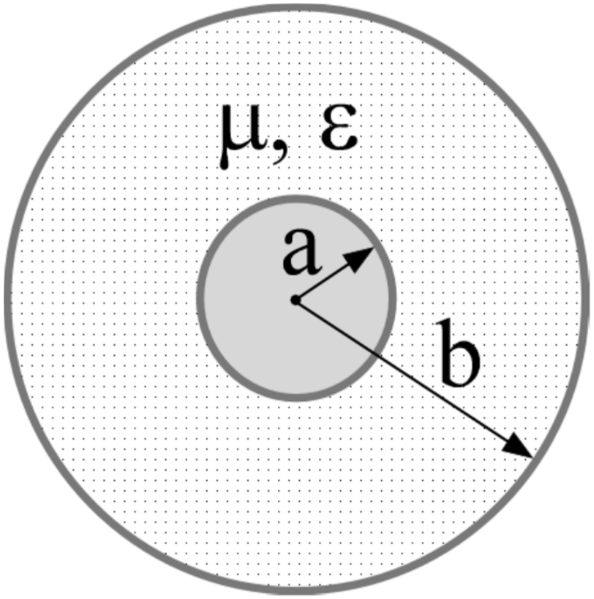
$$BW = f_{cTM_{01}} - f_{cTE_{11}} = \frac{c}{2\pi a}\left(2.4049 - 1.8412\right) = 10GHz$$

From which we find *a* = 0.269 cm

So that

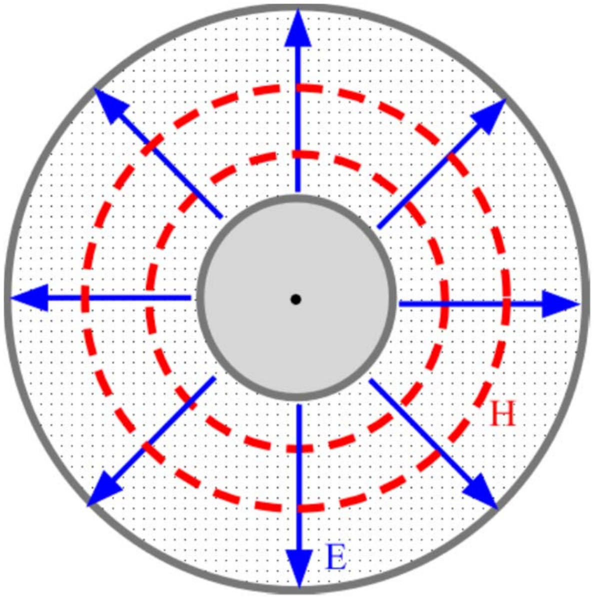
$$f_{cTE_{11}} = 32.7\ GHz\ and\ \ f_{cTM_{11}} = 42.76\ GHz$$

Coaxial Waveguide



- Most common two-conductor transmission system
- Dielectric filling in most microwave applications is polyethylene or Teflon

Coaxial Waveguide – TEM Mode



- Two-conductor system ➔ Dominant mode is TEM
- Tangential E-field and normal H field must be 0 in conductor surfaces

$$E_\phi = 0 \text{ and } H_r = 0 \text{ at } r = a, b$$

Coaxial Waveguide – TEM Mode

TEM solution can exist only with

$$E=\hat{r}E_r\left(r,z\right)\quad\text{and}\quad H=\hat{\phi}H_\phi\left(r,z\right)$$

with no ϕ dependence because of azimuthal symmetry

we get

$$-\frac{\partial H_\phi}{\partial z}=j\omega E_r\rightarrow j\beta H_\phi^o\left(r\right)=j\omega\varepsilon E_r^o\left(r\right)$$

$$-\frac{1}{r}H_\phi+\frac{\partial H_\phi}{\partial r}=0\rightarrow-\frac{1}{r}H_\phi^o\left(r\right)+\frac{\partial H_\phi^o}{\partial r}=0$$

Where propagation in *z* direction is assumed.

Coaxial Waveguide – TEM Mode

We get

$$\mathbf{H} = \hat{\phi}\frac{H_o}{r}e^{-j\beta z} \qquad \mathbf{E} = \hat{r}\frac{H_o\eta}{r}e^{-j\beta z}$$

where H_o is a constant. No cutoff condition for TEM mode.

The voltage between the two conductors is given by

$$V(z) = -\eta H_o \ln(b/a)e^{-j\beta z}$$

The current in the inner conductor is given by

$$I(z) = 2\pi H_o e^{-j\beta z}$$

The characteristic impedance Z_o is thus given by

$$Z_o = \eta\frac{\ln(b/a)}{2\pi}$$

Coaxial Waveguide – TE and TM Modes

TE and TM modes may also exist in addition to TEM. In a coaxial line, they are generally undesirable.

For TM modes, we have:

$$E_z^o(r,\phi)=\left[C_3 J_n(hr)+C_4 Y_n(hr)\right]\cos(n\phi)$$

For TE modes, we have:

$$H_z^o(r,\phi)=\left[C_3^{'} J_n(hr)+C_4^{'} Y_n(hr)\right]\cos(n\phi)$$

With boundary conditions at *r* =*a, b* of

$$E_z(r,\phi)=0 \quad \text{for TM modes}$$

$$\frac{\partial H_z}{\partial r}=0 \quad \text{for TE modes}$$

Coaxial Waveguide – TE and TM Modes

These conditions lead to

$$J_n(ha)Y_n(hb) = J_n(hb)Y_n(ha) \quad \text{for TM modes}$$

$$J_n^{'}(ha)Y_n^{'}(hb) = J_n^{'}(hb)Y_n^{'}(ha) \quad \text{for TE modes}$$

Solutions of these transcendental equations determine the eigenvalues of *h* for given *a, b*. As in the circular waveguide case, the modes for coaxial waveguide are denoted TE_{nl} and TM_{nl}.

Coaxial Waveguide – TE and TM Modes

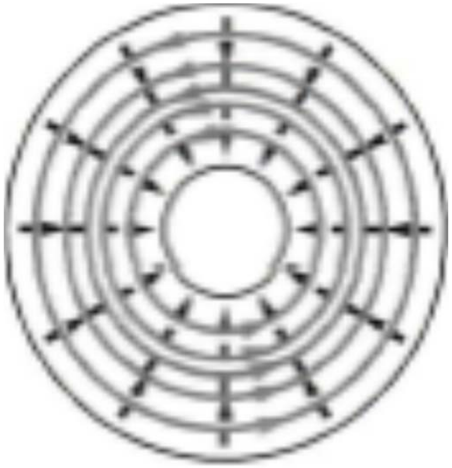
The mode with the lowest cutoff frequency is the TE_{11} mode for which the eigenvalue *h* is approximated as:

$$h = \frac{2}{a+b}$$

The cutoff frequency and cutoff wavelength are given by

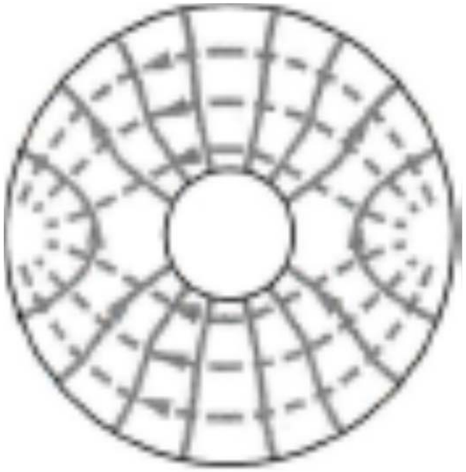
$$\lambda_{c11} = \frac{2\pi}{h} \simeq \pi(a+b) \quad \text{and} \quad f_{c11} \simeq \frac{1}{\pi(a+b)\sqrt{\mu\varepsilon}}$$

Coaxial Waveguide – TE and TM Modes

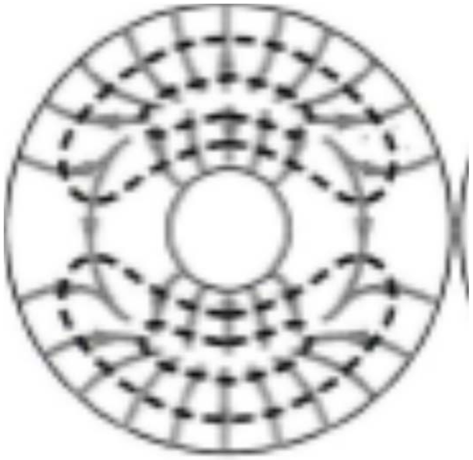


TE$_{01}$

TM$_{01}$



TE$_{11}$



TM$_{11}$

See Reference [3].

References

[1]. **C. S. Lee, S. W. Lee, and S. L. Chuang**, "Plot of modal field distribution in rectangular and circular waveguides", *IEEE Trans. Microwave Theory and Techniques*, 33(3), pp. 271-274, March 1985.

[2]. **J. H. Bryant,** "Coaxial transmission lines, related two-conductor transmission lines, connectors, and components: A U.S. historical perspective", *IEEE Trans. Microwave Theory and Techniques*, 32(9), pp. 970-983, September 1984.

[3]. **H. A. Atwater**, "*Introduction to Microwave Theory*", p. 76, McGraw-Hill, New York, 1962.

[4]. **N. Marcuvitz**, "*Waveguide Handbook*", IEEE Press, Piscataway, New Jersey, 1986.

[5]. **S. Ramo, J. R. Whinnery, and T. Van Duzer**, "*Fields and Waves in Communication Electronics*", John Wiley & Sons, New York, 1994.

[6]. **U. S. Inan and A. S. Inan**, "*Electromagnetice Waves*", Prentice Hall, 2000.