ECE 350

Rectangular Waveguides

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Waveguide

Maxwell’s Equations  \( \nabla^2 \mathbf{E} + \omega^2 \mu \varepsilon \mathbf{E} = 0 \)

\[
\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu \varepsilon E_x
\]

\[
\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu \varepsilon E_y
\]

\[
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu \varepsilon E_z
\]
TE Modes

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set $E_z = 0$. This defines the TE modes.

From $\nabla \times E = -\mu \frac{\partial H}{\partial t}$, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \implies +j\beta_z E_y = -j\omega \mu H_x \quad (1)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \implies -j\beta_z E_x = -j\omega \mu H_y \quad (2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \implies \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega \mu H_z \quad (3)$$
TE Modes

From $\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$, we get

$$j\omega\varepsilon\mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \Rightarrow \frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega\varepsilon E_x \quad (4)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \Rightarrow -j\beta_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad (5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad (6)$$

We want to express all quantities in terms of $H_z$. 
TE Modes

From (2), we have \( H_y = \frac{\beta_z E_x}{\omega \mu} \)

in (4) \( \frac{\partial H_z}{\partial y} + j \beta_z^2 \frac{E_x}{\omega \mu} = j \omega \varepsilon E_x \)

Solving for \( E_x \)

\[ E_x = \frac{j \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial y} \]

From (1) \( H_x = \frac{-\beta_z E_y}{\omega \mu} \)

in (5) \( j \frac{\beta_z^2 E_y}{\omega \mu} - \frac{\partial H_z}{\partial x} = j \omega \varepsilon E_y \)

so that \( E_y = \frac{-j \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial x} \)
TE Modes

\[ H_y = \frac{j \beta_z}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial y} \]

\[ H_x = \frac{j \beta_z}{\beta_z^2 - \omega^2 \mu \varepsilon} \frac{\partial H_z}{\partial x} \]

\[ E_z = 0 \]

Combining solutions for \( E_x \) and \( E_y \) into (3) gives

\[ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] H_z \quad (\mathbf{Y}) \]
If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

\[
\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] H_z \quad (Y)
\]
TE Modes

The general solution for TE modes with $E_z=0$ is obtained from (¥)

$$H_z = e^{-j\beta z} \left[ Ae^{-j\beta x} + Be^{+j\beta x} \right] \left[ Ce^{-j\beta y} + De^{+j\beta y} \right]$$

$$E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \epsilon} e^{-j\beta z} \left[ -Ae^{-j\beta x} + Be^{+j\beta x} \right] \left[ Ce^{-j\beta y} + De^{+j\beta y} \right]$$

$$E_x = \frac{-\beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \epsilon} e^{-j\beta z} \left[ Ae^{-j\beta x} + Be^{+j\beta x} \right] \left[ -Ce^{-j\beta y} + De^{+j\beta y} \right]$$

At $y=0$, $E_x=0$ which leads to $C=D$

At $x=0$, $E_y=0$ which leads to $A=B$
TE Modes

\[ H_z = H_0 e^{-j\beta_z z} \cos \beta_x x \cos \beta_y y \quad (§) \]

\[ E_y = \frac{j \beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z} \sin \beta_x x \cos \beta_y y \]

\[ E_x = \frac{-j \beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z} \cos \beta_x x \sin \beta_y y \]

At \( x=a \), \( E_y=0 \) which leads to \( \beta_x = \frac{m\pi}{a} \)

At \( y=b \), \( E_x=0 \) which leads to \( \beta_y = \frac{n\pi}{b} \)

The general solution for TE modes with \( E_z=0 \) is
Dispersion Relation

The dispersion relation is obtained by placing (§) in (¥)
\[ \beta_z^2 + \beta_x^2 + \beta_y^2 = \omega^2 \mu \varepsilon \quad (23) \]
\[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \beta_z^2 = \omega^2 \mu \varepsilon \quad (24) \]
\[ \beta_z = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \quad (25) \]

The guidance condition is
\[ \omega^2 \mu \varepsilon > \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \quad (26) \]
Guidance Condition

or $f > f_c$ where $f_c$ is the cutoff frequency of the $\text{TE}_{mn}$ mode given by the relation

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

The $\text{TE}_{mn}$ mode will not propagate unless $f$ is greater than $f_c$.

Obviously, different modes will have different cutoff frequencies.
TM Mode

The transverse magnetic modes for a general waveguide are obtained by assuming $H_z = 0$. By duality with the TE modes, we have

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = \left[ \beta_z^2 - \omega^2 \mu \varepsilon \right] E_z$$

$$E_z = e^{-j\beta_z z} \left[ Ae^{-j\beta_x x} + Be^{j\beta_x x} \right] \left[ Ce^{-j\beta_y y} + De^{j\beta_y y} \right]$$
TM Mode

The boundary conditions are

At $x=0$, $E_z=0$ which leads to $A=-B$

At $y=0$, $E_z=0$ which leads to $C=-D$

At $x=a$, $E_z=0$ which leads to $\beta_x = \frac{m\pi}{a}$

At $y=b$, $E_z=0$ which leads to $\beta_y = \frac{n\pi}{b}$
TM and TE Modes

so that the generating equation for the $TM_{mn}$ modes is

$$E_z = E_o e^{-j\beta_z} \sin \beta_x x \sin \beta_y y$$

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

For additional information on the field equations see Rao (6th Edition), page 607, Table 9.1.
TE and TM Modes

There is no $\text{TE}_{00}$ mode

There are no $\text{TM}_{m0}$ or $\text{TM}_{0n}$ modes

The first TE mode is the $\text{TE}_{10}$ mode

The first TM mode is the $\text{TM}_{11}$ mode
Impedance of a Waveguide

For a TE mode, we define the transverse impedance as

\[ \eta_{gTE} = \frac{-E_y}{H_x} = \frac{E_x}{H_y} = \frac{\omega \mu}{\beta_z} \]

From the relationship for \( \beta_z \) and using

we get

\[ f_c^2 = \frac{1}{4\mu\varepsilon} \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right] \]

\[ \eta_{gTE} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}} \]

where \( \eta \) is the intrinsic impedance

\[ \eta = \sqrt{\frac{\mu}{\varepsilon}} \]
Impedance of a Waveguide

Analogously, for TM modes, it can be shown that

\[ \eta_{g_{TM}} = \eta \sqrt{1 - \frac{f_c^2}{f^2}} \]
Power Flow in a Waveguide

TE$_{10}$ Mode

The time-average Poynting vector for the TE$_{10}$ mode in a rectangular waveguide is given by

$$\langle P \rangle = \frac{1}{2} \text{Re} \left[ E \times H^* \right] = \hat{z} \frac{|E_o|^2}{2} \frac{\beta_z}{\omega \mu} \sin^2 \frac{\pi x}{a}$$

$$\langle \text{Power} \rangle = \int_0^a \int_0^b |E_o|^2 \frac{\beta_z}{2 \omega \mu} \sin^2 \frac{\pi x}{a} dx dy$$

$$\langle \text{Power} \rangle = \frac{|E_o|^2}{4} \frac{\beta_z ab}{\omega \mu} = \frac{|E_o|^2}{4} \frac{ab}{\eta_{gTE_{10}}}$$

The time-average power flow in a waveguide is proportional to its cross-section area.