ECE 453
Wireless Communication Systems

The Smith Chart

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TL Equations

Voltage

$$V(z) = V_+ e^{-j\beta z}\left[1+\Gamma_R e^{+2j\beta z}\right]$$

Current

$$I(z) = \frac{V_+}{Z_o} e^{-j\beta z}\left[1-\Gamma_R e^{+2j\beta z}\right]$$

Impedance Transformation ➔ $$Z(-l) = Z_o\left[\frac{Z_R + jZ_o \tan\beta l}{Z_o + jZ_R \tan\beta l}\right]$$

Reflection Coefficient Transformation ➔ $$\Gamma(-l) = \Gamma_R e^{-2j\beta l}$$

Reflection Coefficient – to Impedance ➔ $$Z(z) = Z_o \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

Impedance to Reflection Coefficient ➔ $$\Gamma(z) = \frac{Z(z) - Z_o}{Z(z) + Z_o}$$

Derivation of the Smith Chart

The relationship between impedance and reflection coefficient is given by:

$$Z(z) = Z_o\left[\frac{1+\Gamma(z)}{1-\Gamma(z)}\right]$$

where Z_o is the characteristic impedance of the system. The normalized impedance is

$$Z_n(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)} = \frac{1+\Gamma}{1-\Gamma}$$

The reflection coefficient and the normalized impedance have the form:

$$\Gamma = \Gamma_r + j\Gamma_i \quad \textbf{and} \quad Z_n = r + jx$$

Derivation of the Smith Chart

Therefore

$$r + jx = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i} = \frac{\left[(1+\Gamma_r)+j\Gamma_i\right]\left[(1-\Gamma_r)+j\Gamma_i\right]}{(1-\Gamma_r)^2+\Gamma_i^2}$$

Separating real and imaginary components,

$$r + jx = \frac{1-\Gamma_r^2 + j\Gamma_i(1+\Gamma_r) + j\Gamma_i(1-\Gamma_r) - \Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2}$$

Isolating the real part from both sides

$$r = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2}$$

Derivation of the Smith Chart

Multiplying through by the denominator,

$$r\left[1+\Gamma_r^2-2\Gamma_r+\Gamma_i^2\right]=1-\Gamma_r^2-\Gamma_i$$

$$\Gamma_r^2(r+1)+\Gamma_i^2(r+1)-2r\Gamma_r=1-r$$

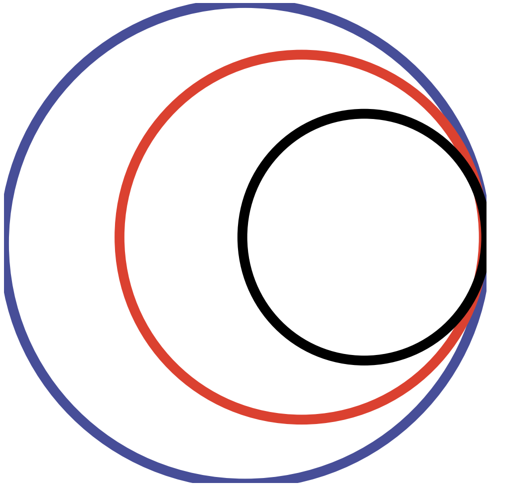
$$\Gamma_r^2+\Gamma_i^2-\frac{2r\Gamma_r}{1+r}+\frac{r^2}{(1+r)^2}=\frac{1-r}{1+r}+\frac{r^2}{(1+r)^2}$$

Completing the square

$$\Gamma_r^2+\Gamma_i^2-\frac{2r\Gamma_r}{1+r}=\frac{1-r}{1+r}$$ **or** $$\left(\Gamma_r-\frac{r}{1+r}\right)^2+\Gamma_i^2=\frac{1}{(1+r)^2}$$

Derivation of the Smith Chart

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2}$$



This is the equation of a circle centered at

$$\left(\frac{r}{1+r}, 0\right)$$ and of radius $$\frac{1}{1+r}$$

Equating the imaginary parts gives

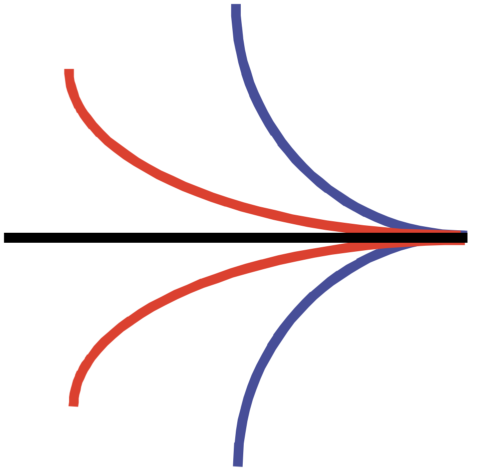
$$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$x\left[1 + \Gamma_r^2 - 2\Gamma_r + \Gamma_i^2\right] = 2\Gamma_i \quad \text{or} \quad \Gamma_r^2 x - 2x\Gamma_r + x\Gamma_i^2 - 2\Gamma_i = -x$$

Derivation of the Smith Chart

$$\Gamma_r^2 - 2\Gamma_r + 1 + \Gamma_i^2 - \frac{2\Gamma_i}{x} + \frac{1}{x^2} = \frac{1}{x^2} - 1 + 1$$

$$\left(\Gamma_r - 1\right)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$



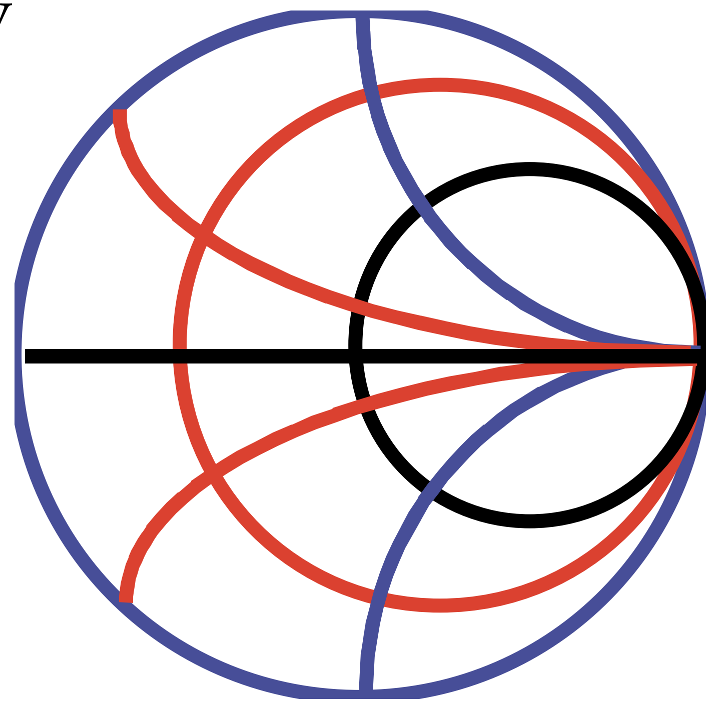
This is the equation of a circle centered at

$\left(1, \frac{1}{x}\right)$ of radius $\frac{1}{x}$

The Smith Chart

The reflection coefficient is given by

$$\Gamma = \frac{Z_n - 1}{Z_n + 1} = \frac{r - 1 + jx}{r + 1 + jx}$$



We also have

$$|\Gamma| = \left[\frac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2} \leq 1$$

$$Z_n = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

$$y = \frac{1}{Z_n} = \frac{1 - \Gamma(z)}{1 + \Gamma(z)}$$

Thus, going from normalized impedance to normalized admittance corresponds to a 180 degree shift.

The Smith Chart

3 ways to move on the Smith chart

- Constant SWR circle → displacement along TL
- Constant resistance (conductance) circle → addition of reactance (susceptance)
- Constant reactance (susceptance) arc → addition of resistance (conductance)

Smith Chart Example

Results of several different experiments are plotted on a Smith chart. Each experiment measured the input reflection coefficient from a low frequency (denoted by a circle) to a high frequency (denoted by a square) of a one-port. Determine the load that was measured. The loads that were measured were one of those shown on the table below.

You should make no assumptions about how low the low frequency was, nor about how high the high frequency was. For each of the measurements below indicate the load using the load identifier above (e.g., i, ii, etc.) There may be more than one correct answer.

(a) What type of load gives rise to the refection coefficient indicated by curve A?

(b) What type of load gives rise to the refection coefficient indicated by curve B?

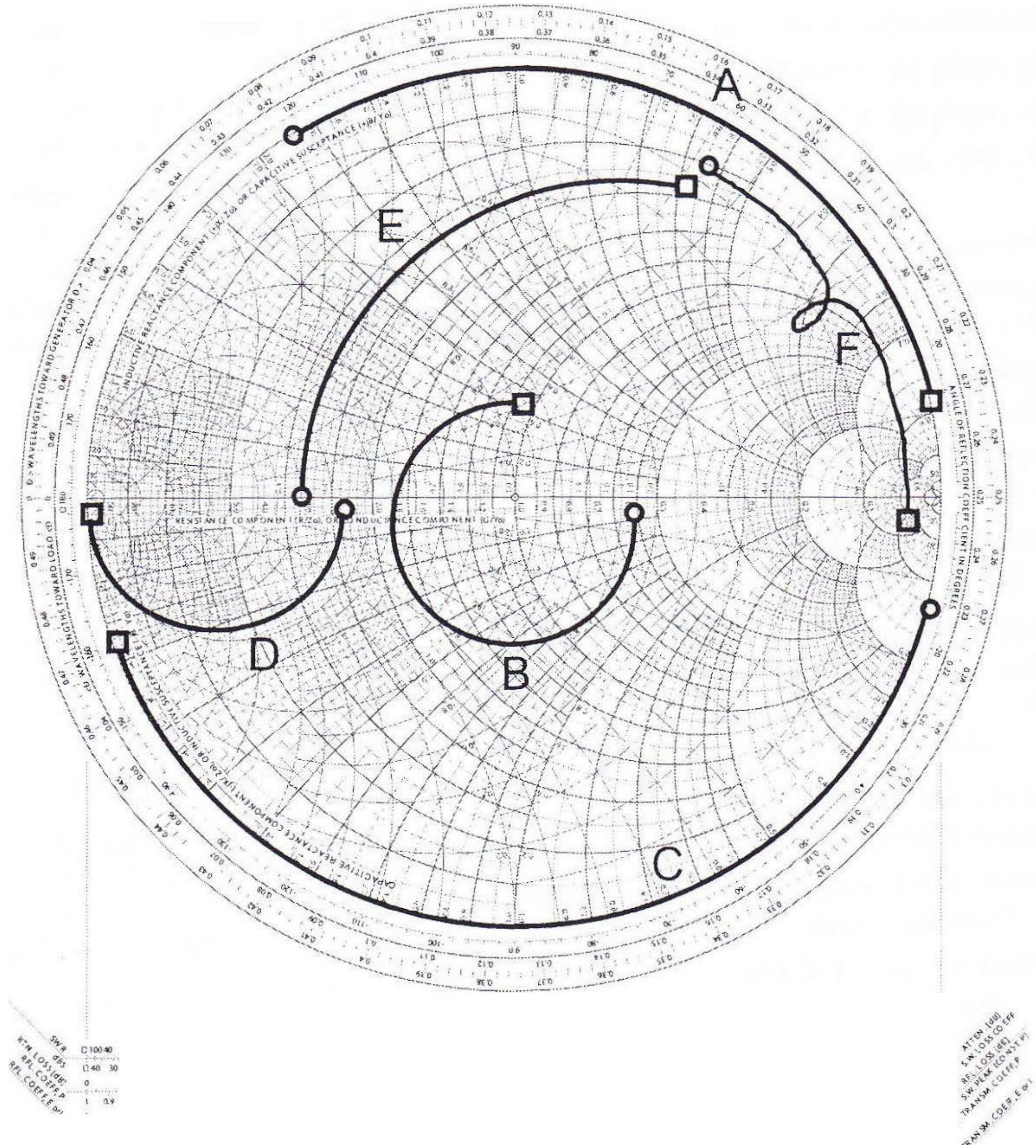
(c) What type of load gives rise to the refection coefficient indicated by curve C?

(d) What type of load gives rise to the refection coefficient indicated by curve D?

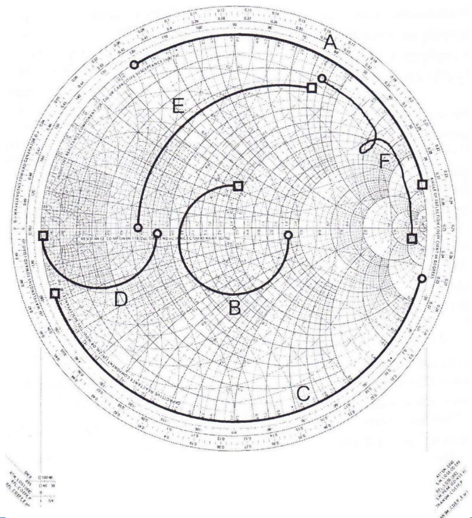
(e) What type of load gives rise to the refection coefficient indicated by curve E?

(f) What type of load gives rise to the refection coefficient indicated by curve F?

Smith Chart Example



Smith Chart Example



Load	Description
i	An inductor
ii	A capacitor
iii	A reactive load at the end of a transmission line
iv	A resistive load at the end of a transmission line
v	A parallel connection of an inductor, a resistor, and a capacitor going through resonance and with a transmission line offset
vi	A series connection of a resistor, an inductor and a capacitor going through resonance and with a transmission line offset
vii	A series resistor and inductor
viii	A shunt connection of a resistor and a capacitor

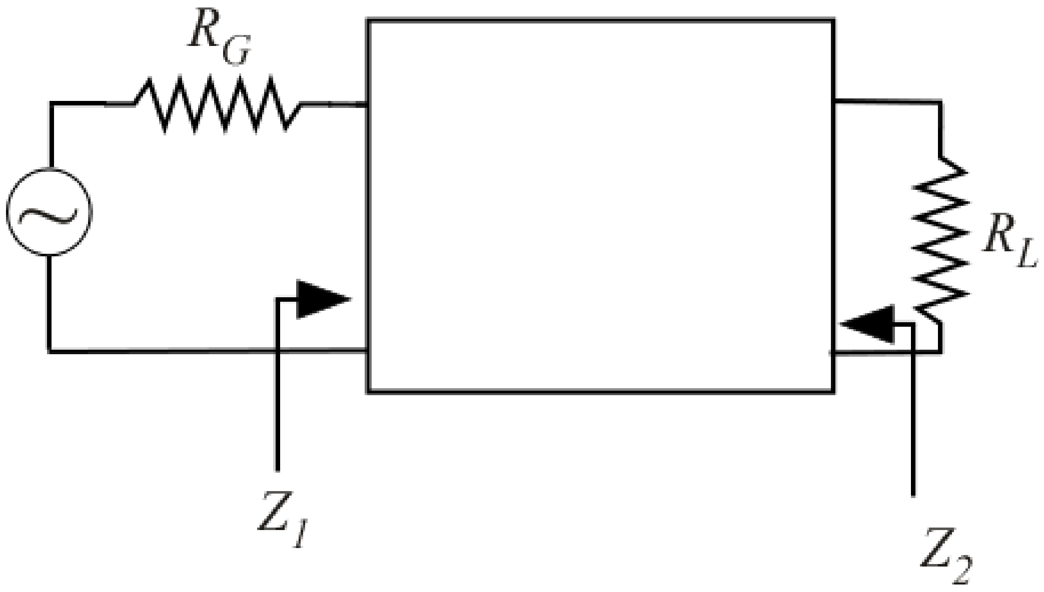
Smith Chart Example

Solutions

Load	Description
A	An inductor – or a reactive load at the end of a transmission line (i and iii)
B	A resistive load at the end of a transmission line (iv)
C	A capacitor (ii)
D	A shunt connection of a resistor and a capacitor (viii)
E	A series resistor and inductor (vii)
F	A series connection of a resistor, an inductor and a capacitor going through resonance and with a transmission line offset (vi)

Smith Chart Example

Develop a two-element matching network to match a source with an impedance of R_G = 25 Ω to a load R_L = 200 Ω

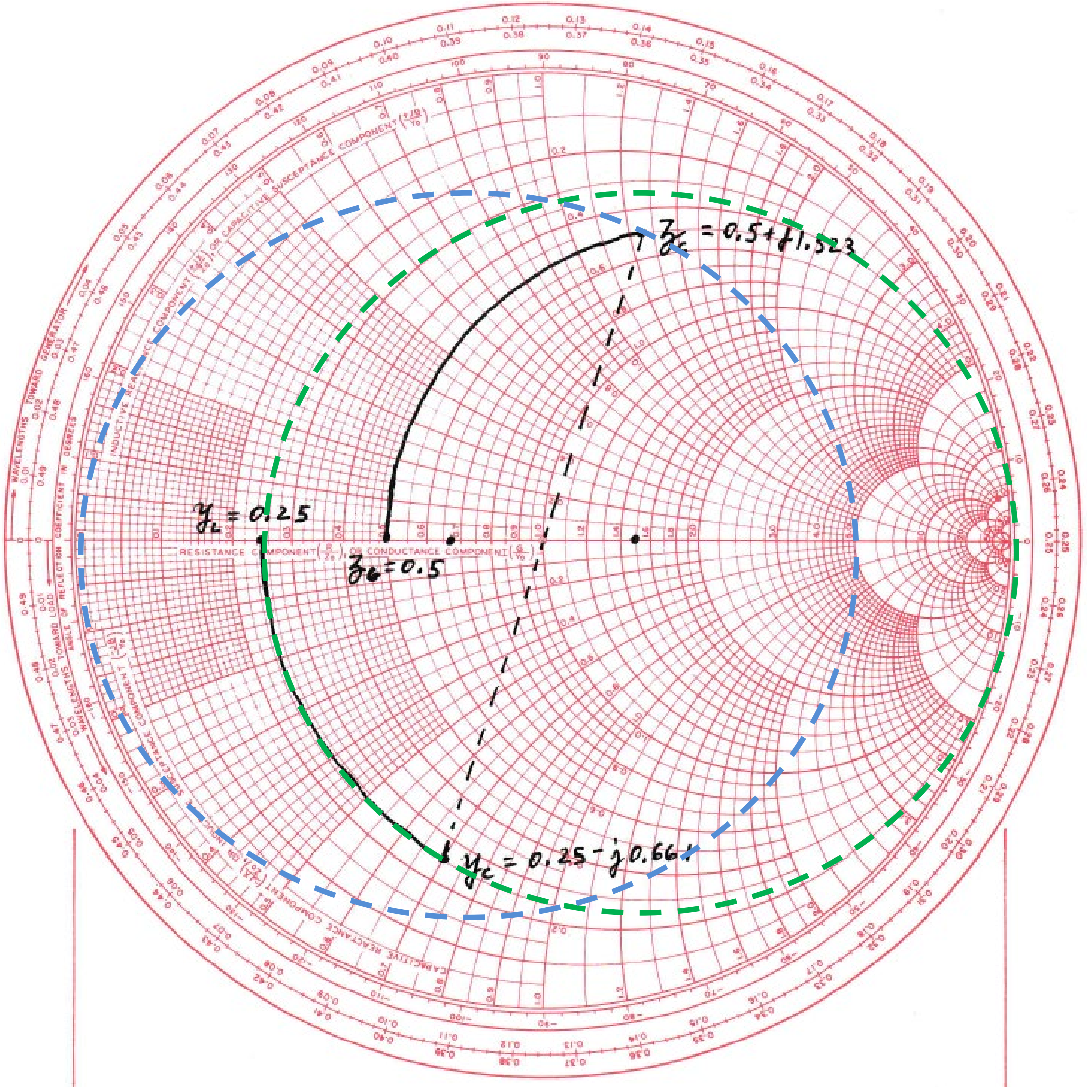


Smith Chart Example

Solution (from source to load)

1. Normalize to 50 Ω. Then $z_G = r_G = 0.5$ and $z_L = r_L = 4.0$
2. Enter Smith chart at 0.5+j0
3. Identify circle $r = 0.5$
4. Normalized admittance at load is 0.25
5. Identify $g = 0.25$ circle ($g = 1/r$)
6. Find center of $g = 0.25$ circle → (0.2,0)
7. Rotate center by 180 degrees
8. From new center draw circle of radius 0.25/(1+0.25) which intersects $r = 0.5$ circle at $0.5+j1.323 = z_C$.
9. Rotate z_C by 180 degrees. By construction, it will land on $g = 0.25$ circle at $y_C = 0.25-j0.661$
10. Move along $g = 0.25$ circle until intersection with horizontal axis.

Smith Chart Example



Smith Chart Example

Change in normalized reactance from source to point C

$$x_S = x_C - x_G = 1.323 - 0 = 1.323$$

Reactance value

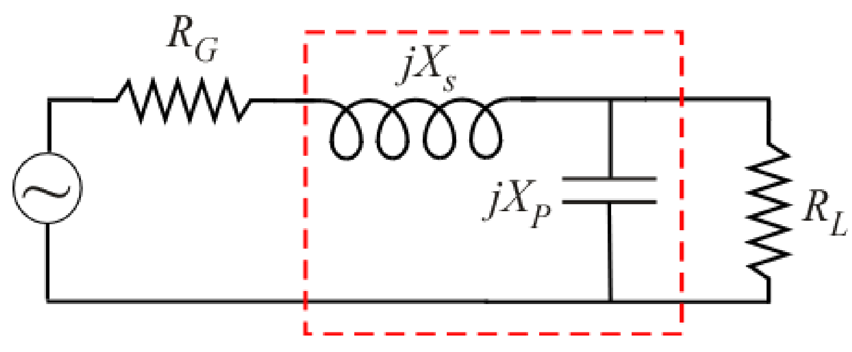
$$X_S = x_S \times Z_o = 1.323 \times 50 = 66.1\Omega$$

Change in normalized susceptance from point C to load

$$b_P = b_L - b_C = 0 - (-0.661) = 0.661$$

Susceptance value

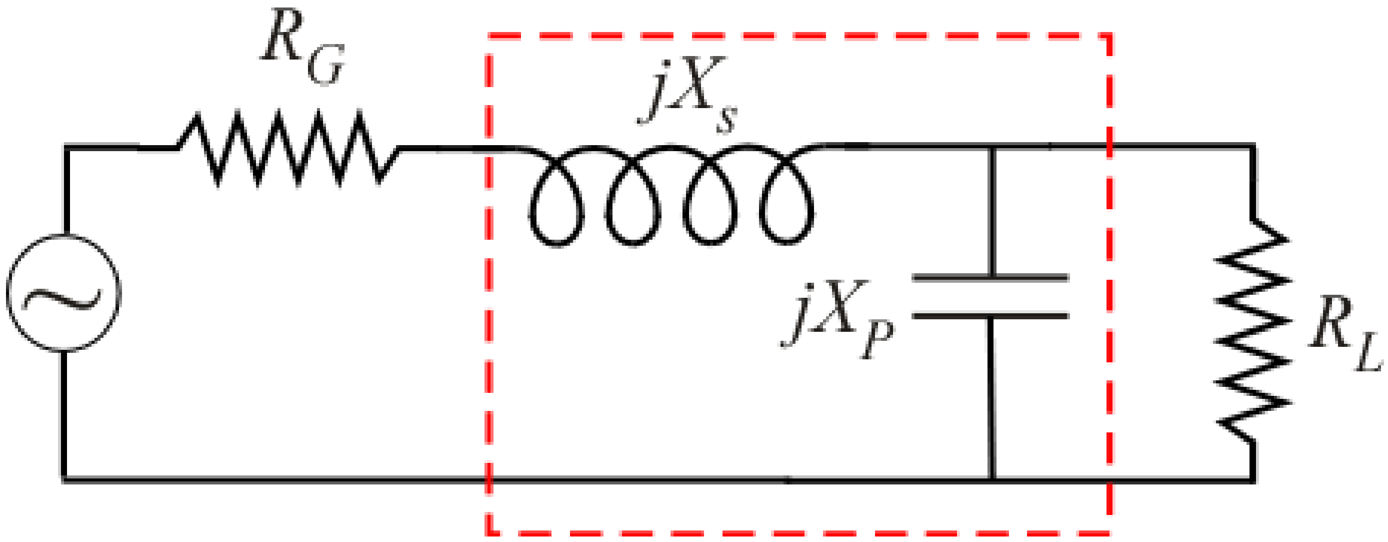
$$B_P = \frac{b_P}{Z_o} = \frac{0.661}{50} = 132\ mS$$



Reactance value (in parallel)

$$X_P = -1 / B_P = -75.6\Omega$$

Smith Chart Example



$$X_S = 66.1\Omega$$

$$X_P = -75.6\Omega$$