

# **ECE 453**

# **Wireless Communication Systems**

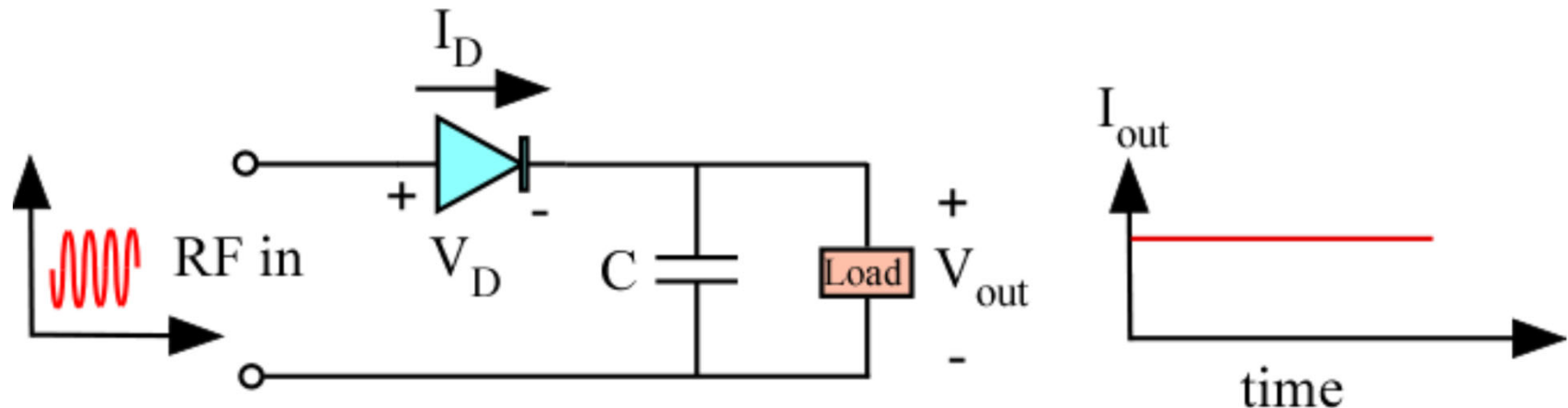
  

## **Nonlinearities**

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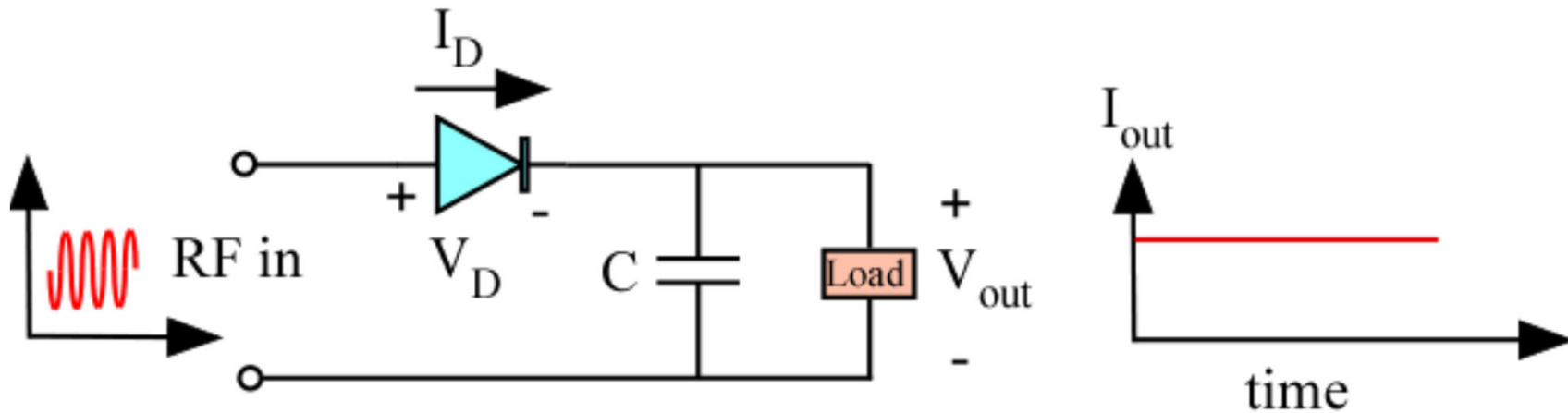
# Square-Law Detector

- Detector will receive RF at input and will produce a DC voltage proportional to the magnitude of the RF input



In order to operate properly, diode current must remain in the square-law region

# Square-Law Detector

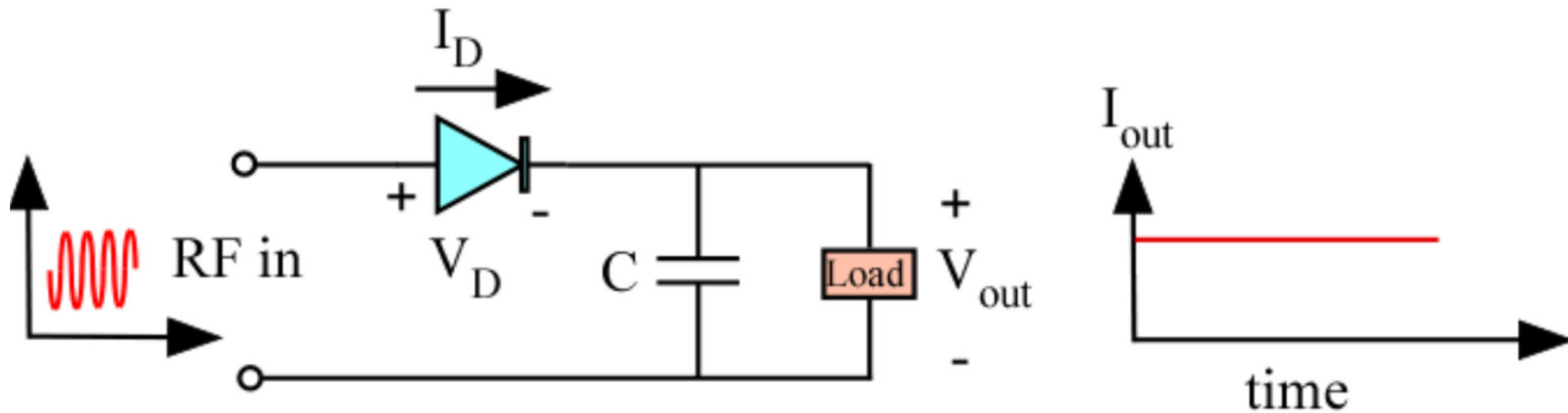


$$I_D = I_S \left( e^{\frac{qV_D}{kT}} - 1 \right) = I_S \left( e^{\frac{V_D}{V_T}} - 1 \right)$$

**Expanding**

$$I_D \approx I_S \left[ \frac{V_D}{V_T} + \frac{1}{2} \left( \frac{V_D}{V_T} \right)^2 + \frac{1}{6} \left( \frac{V_D}{V_T} \right)^3 + \dots \right]$$

# Square-Law Detector



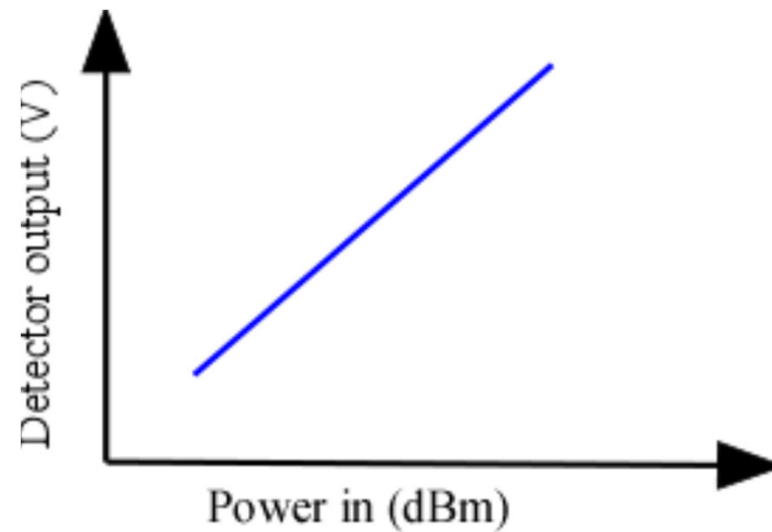
If  $V_D$  is small  $I_D \approx I_S \left[ \frac{V_D}{V_T} + \frac{1}{2} \left( \frac{V_D}{V_T} \right)^2 + \frac{1}{6} \left( \frac{V_D}{V_T} \right)^3 + \dots \right]$   $I_{out} = aV_D$  where  $a$  is some constant

If  $V_D$  is larger  $I_D \approx I_S \left[ \frac{V_D}{V_T} + \frac{1}{2} \left( \frac{V_D}{V_T} \right)^2 + \frac{1}{6} \left( \frac{V_D}{V_T} \right)^3 + \dots \right]$   $I_{out} = bV_D^2$  where  $b$  is some constant

If  $V_D$  is even larger  $I_D \approx I_S \left[ \frac{V_D}{V_T} + \frac{1}{2} \left( \frac{V_D}{V_T} \right)^2 + \frac{1}{6} \left( \frac{V_D}{V_T} \right)^3 + \dots \right]$   $I_{out} = cV_D^3$  where  $c$  is some constant

# Square-Law Detector

**GOAL**: Keep detector operating in square-law range. Then,  $I_{out}$  is proportional to  $V_D^2$  and thus proportional to the power input.



# Power Series Model

Input-output relationship known as:

$$y = f(x)$$

can be expressed as:

$$y = k_1 x + k_2 x^2 + k_3 x^3 + \dots$$

where

$$k_i = \left. \frac{d^i f}{dx^i} \right|_{x=x_q}$$

# BJT Power Series Model

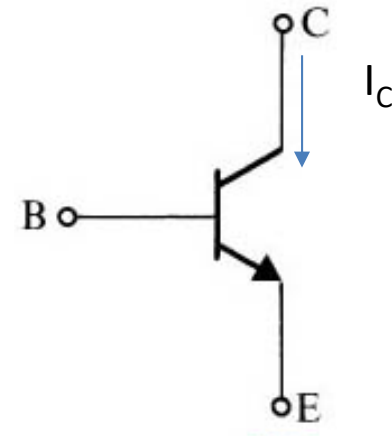
## Collector current

$$I_C = I_S e^{V_{be}/V_T}$$

## Base-emitter junction voltage

$$V_{be} = V_{DC} + v_{be}$$

$$I_C = I_S e^{V_{DC}/V_T} e^{v_{be}/V_T}$$



## Collector current in power series form

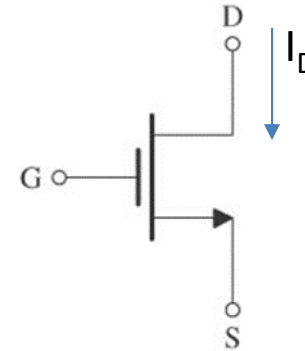
$$I_C = I_S e^{V_{DC}/V_T} \left[ 1 + \frac{v_{be}}{V_T} + \frac{1}{2} \left( \frac{v_{be}}{V_T} \right)^2 + \frac{1}{6} \left( \frac{v_{be}}{V_T} \right)^3 + \dots \right]$$

# Power Series Model for MOSFET

## MOSFET drain current

$$I_D = \mu_n C_{ox} \frac{W}{2L} (V_{GS} - V_T)^2$$

$$I_D = \mu_n C_{ox} \frac{W}{2L} (V_{GS}^2 - 2V_{GS}V_T + V_T^2)$$



**No third order or higher terms → idealized square-law characteristic**

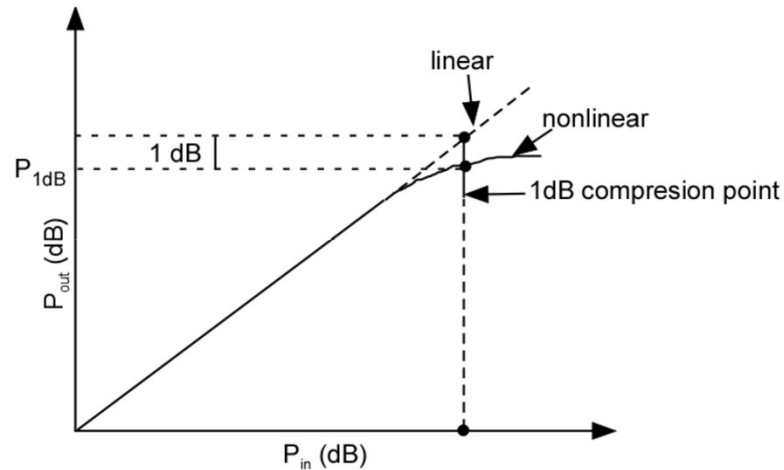


# Distortion and Nonlinearities

Distortion occurs when the output signal from an amplifier approaches the extremes of the load line and the output is no longer an exact replica of the input.

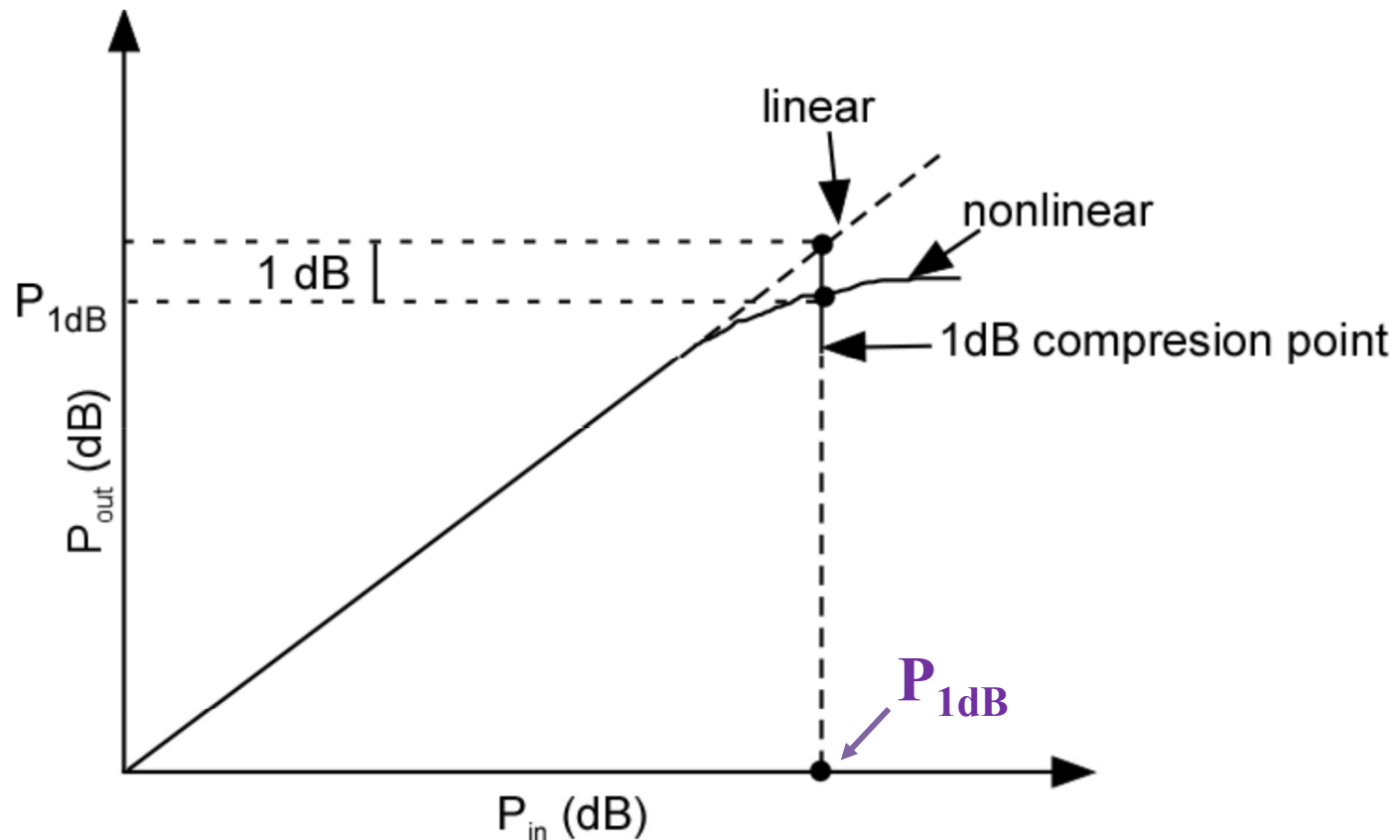
The ideal amplifier follows a linear relationship. Distortion leads to a deviation from this linear relationship

# Distortion and Nonlinearities



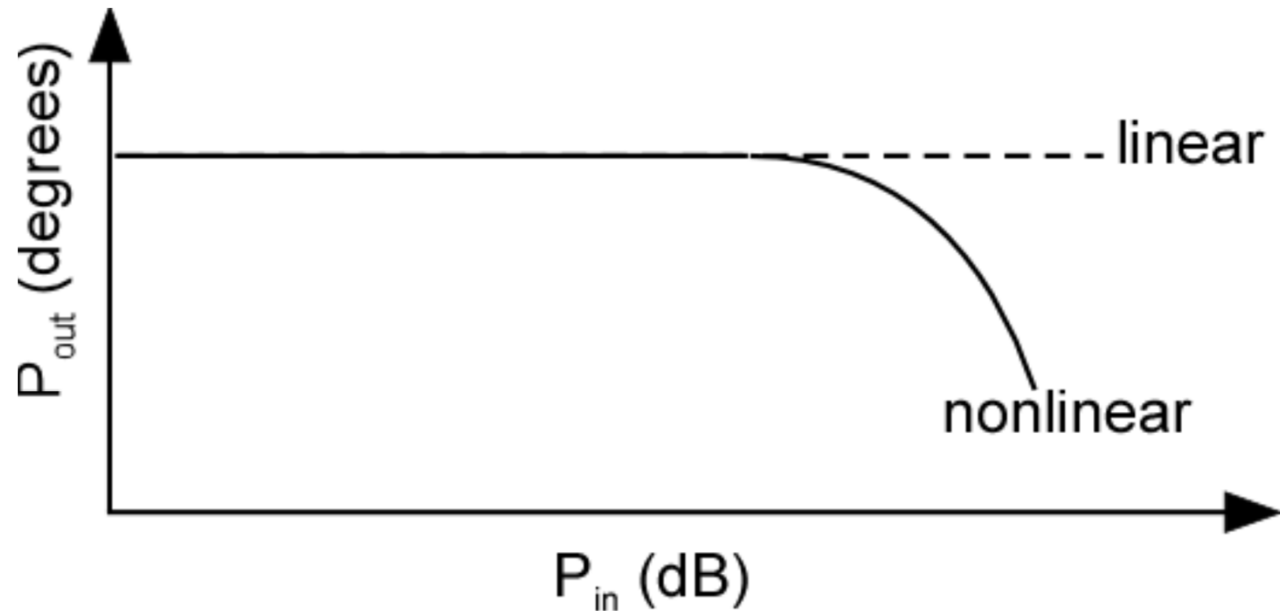
The *1dB compression point* is the point where the difference between the extrapolated linear response exceeds the actual gain by 1dB.  $P_{1dB}$  is the power output at the 1dB compression point and is the most important metric of distortion.

# Distortion and Nonlinearities



## AM-AM Distortion

# Distortion and Nonlinearities



## AM-PM Distortion

# Distortion and Nonlinearities

AM-AM distortion is generally more significant

Distortion is a result of nonlinearities in the response of an amplifier. One possible model for the response is:

$$V_{out} = aV_{in} + bV_{in}^2 + cV_{in}^3 + \dots$$

When the input signal is large enough the higher order terms become significant

# Distortion and Nonlinearities

If the input is a sine wave, the output will contain harmonics of the input. That is the output will have frequency components that are integer multiples of the fundamental (CW) sine wave input signal

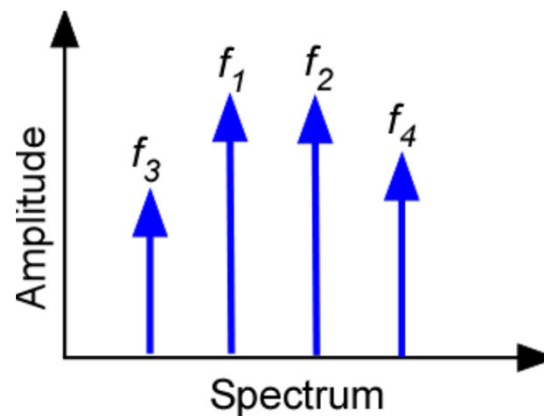
When the input signal is a two-tone signal (modulated signal), additional tones will appear at the output and we say that the distortion produces intermodulation products (IMP)

# Distortion and Nonlinearities

If  $f_1$  and  $f_2$  are the frequencies associated with the two signals, the output will have components at  $f_3$  and  $f_4$  where

$$f_3 = 2f_1 - f_2 \quad \text{and} \quad f_4 = 2f_2 - f_1$$

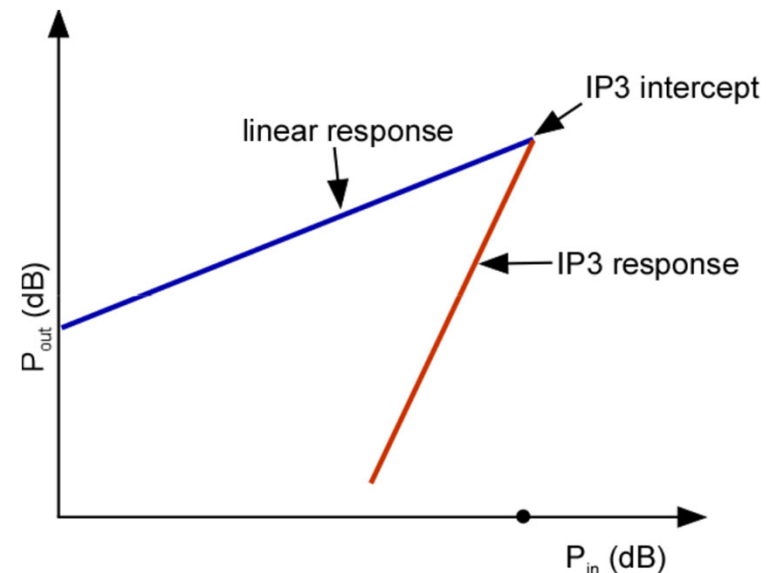
are known as the lower and upper IM3 respectively.



# Distortion and Intermodulation

At low power, before compression the fundamental has a response with 1:1 slope with respect to the input

The IP3 response varies as the *cube* of the level of input tones. The IP3 has a 3:1 logarithmic slope with respect to the input.



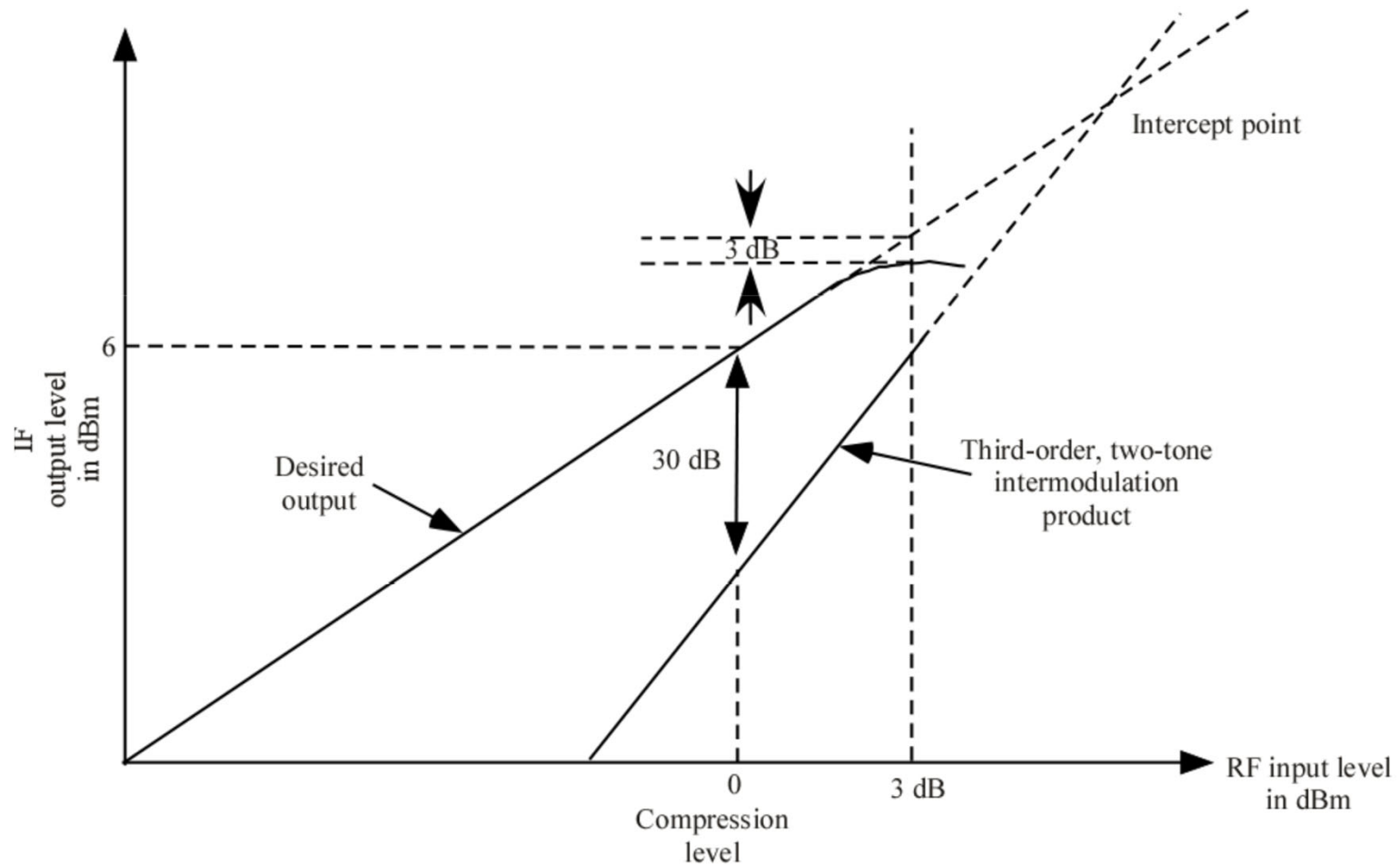


# Distortion and Intermodulation

The point of intersection of the extrapolated linear output (of power  $P_o$ ) and third order (IP3) of power  $P_{IP3}$  is called the third-order intercept point which is a key parameter

- Harmonics can be filtered out by filters
- Intermodulation distortion cannot be filtered out because they are in the main passband

# Distortion and Intermodulation



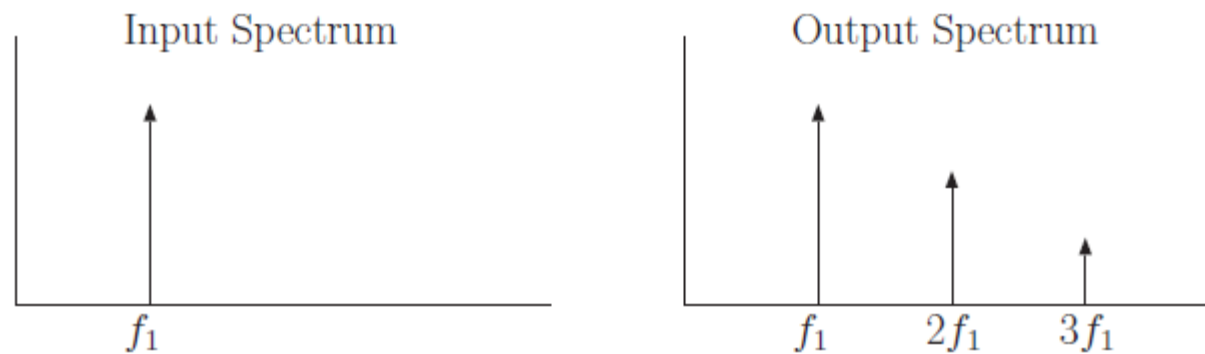
# Single-Tone Excitation

$$v_o = k_1 v_i + k_2 v_i^2 + k_3 v_i^3$$

$$v_i(t) = a_1 \cos \omega_1 t$$

$$v_o(t) = \frac{1}{2} k_2 a_1^2 + \left( k_1 a_1 + \frac{3}{4} k_3 a_1^2 \right) \cos \omega_1 t + \\ \frac{1}{2} k_2 a_1^2 \cos 2\omega_1 t + \frac{1}{4} k_3 a_1^3 \cos 3\omega_1 t$$

# Single-Tone Excitation



**Single-tone excitation produces harmonics of input frequency**

# Two-Tone Excitation

$$v_o = k_1 v_i + k_2 v_i^2 + k_3 v_i^3$$

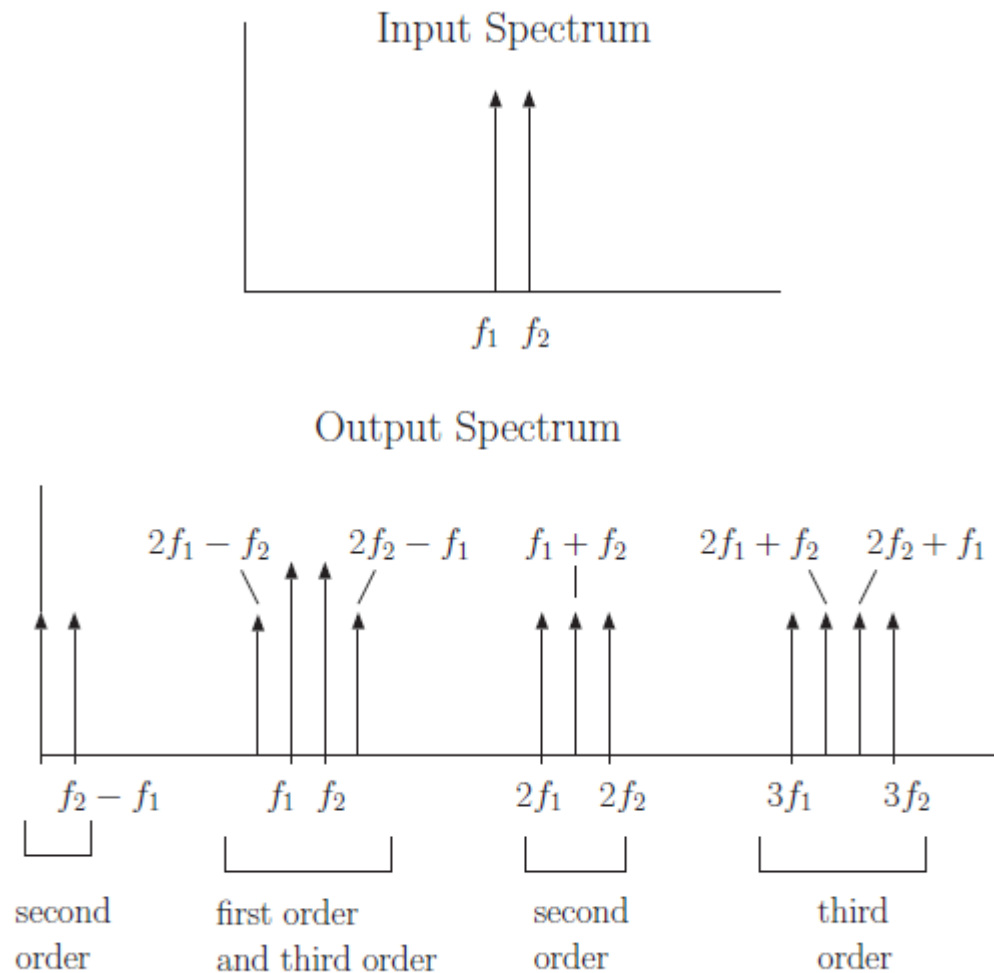
$$v_i(t) = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t$$

$$v_o(t) = k_1 [a_1 \cos \omega_1 t + a_2 \cos \omega_2 t] +$$

$$k_2 \left[ \frac{1}{2} (a_1^2 + a_2^2) + \frac{1}{2} a_1^2 \cos 2\omega_1 t + \frac{1}{2} a_2^2 \cos 2\omega_2 t \right. \\ \left. + a_1 a_2 \cos(\omega_1 + \omega_2)t + a_1 a_2 \cos(\omega_1 - \omega_2)t \right] +$$

$$\frac{1}{4} k_3 \left[ \frac{1}{4} a_1^3 \cos 3\omega_1 t + \frac{1}{4} a_2^3 \cos 3\omega_2 t \right. \\ \left. + \frac{3}{4} a_1 a_2^2 (\cos(2\omega_2 - \omega_1)t + \cos(2\omega_2 + \omega_1)) \right. \\ \left. + \frac{3}{4} a_1^2 a_2 (\cos(2\omega_1 - \omega_2)t + \cos(2\omega_1 + \omega_2)) \right]$$

# Two-Tone Excitation



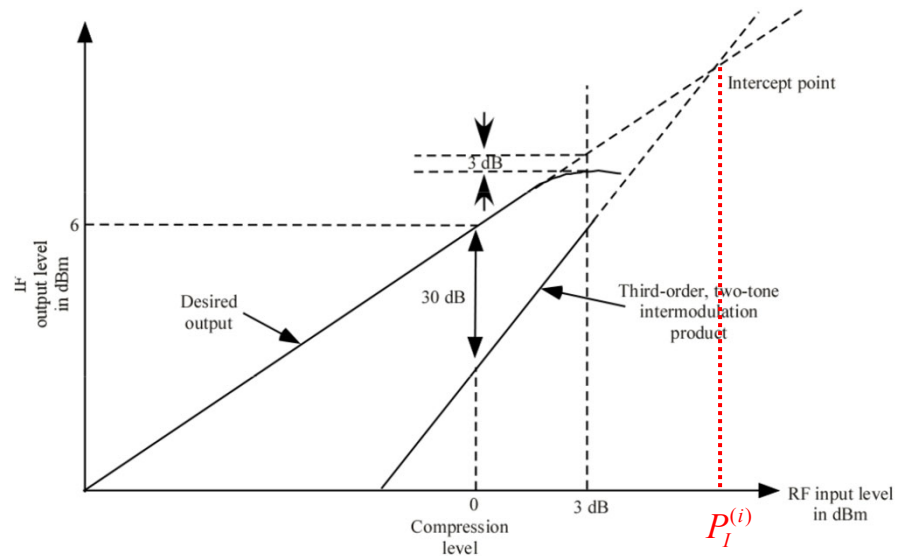
**Two-tone excitation produces intermodulation**

# Intermodulation Ratio (IMR)

$$IMR = \frac{P_{IM}}{P_d} = \left( \frac{P_{in}}{P_I^{(i)}} \right)^2$$

**In dB:**

$$IMR(dB) = 2P_{in}(dBm) - 2P_I^{(i)}(dBm)$$



$P_{in}$  **input power of fundamental components**

$P_d$  **output power of fundamental components**

$P_{IM}$  **output power of in-band 3<sup>rd</sup> – order intermodulation**

$P_I^{(i)}$  **third-order intercept**

# Dynamic Range of Receiving System

Dynamic range is defined as:

$$DR = \frac{\text{Maximum useable input signal power level}}{\text{Minimum useable input signal power level}}$$

$P_{\min}$  → input power for specified  $SNR_{o,\min}$  (MDS)

$P_{\max}$  → two-tone input power (per tone) at which  
SNR of in-band 3<sup>rd</sup> order IM product =  $SNR_{o,\min}$  at  
system output

Can show that  $DR = \frac{2}{3} [P_I^{(i)} - P_{\min}]$

$P_I^{(i)}$  is the third-order intercept



# Mixer Analysis

**Input is described by:**

$$x(t) = |X_1| \cos(\omega_1 t + \phi_1) + |X_2| \cos(\omega_2 t + \phi_2)$$

**In complex notation:**

$$x(t) = \frac{1}{2} \left[ X_1 e^{j\omega_1 t} + X_1^* e^{-j\omega_1 t} + X_2 e^{j\omega_2 t} + X_2^* e^{-j\omega_2 t} \right]$$

**System is such that the output is described by:**

$$y(t) = a_o + a_1 x(t) + a_2 x^2(t) + a_3 x^3(t)$$

# Intermodulation Products

Intermodulation Product	Frequency	Order
$\frac{1}{2}(X_1 X_1^*)$	0	2
$\frac{1}{2}(X_2 X_2^*)$	0	2
$2\frac{1}{2}X_1 = X_1$	$\omega_1$	1
$2(\frac{1}{2})^3 3X_1^2 X_1^* = \frac{3}{4}X_1^2 X_1^*$	$\omega_1$	3
$2(\frac{1}{2})^3 6X_1 X_2 X_2^* = 3(\frac{1}{2})X_1 X_2 X_2^*$	$\omega_1$	3
$2\frac{1}{2}X_2 = X_2$	$\omega_2$	1
$2(\frac{1}{2})^3 3X_2^2 X_2^* = \frac{3}{4}X_2^2 X_2^*$	$\omega_2$	3
$2(\frac{1}{2})^3 6X_2 X_1 X_1^* = 3(\frac{1}{2})X_2 X_1 X_1^*$	$\omega_2$	3
$2(\frac{1}{2})^2 X_1^2 = \frac{1}{2}X_1^2$	$2\omega_1$	2
$2(\frac{1}{2})^2 X_2^2 = \frac{1}{2}X_2^2$	$2\omega_2$	2
$2(\frac{1}{2})^3 X_1^2 = \frac{1}{4}X_1^3$	$3\omega_1$	3
$2(\frac{1}{2})^3 X_2^2 = \frac{1}{4}X_2^3$	$3\omega_2$	3
$2(\frac{1}{2})X_1 X_2 = X_1 X_2$	$\omega_1 + \omega_2$	2
$2(\frac{1}{2})X_1 X_2^* = X_1 X_2^*$	$\omega_1 - \omega_2$	2
$2(\frac{1}{2})^3 3X_1^2 X_2 = \frac{3}{4}X_1^2 X_2$	$2\omega_1 + \omega_2$	3
$2(\frac{1}{2})^3 3X_1^2 X_2^* = \frac{3}{4}X_1^2 X_2^*$	$2\omega_1 - \omega_2$	3
$2(\frac{1}{2})^3 3X_2^2 X_1 = \frac{3}{4}X_2^2 X_1$	$\omega_1 + 2\omega_2$	3
$2(\frac{1}{2})^3 3X_2^2 X_1^* = \frac{3}{4}X_2^2 X_1^*$	$\omega_1 - 2\omega_2$	3

# Volterra Series

A linear causal system with memory can be described by the convolution representation

$$y(t) = \int_{-\infty}^{+\infty} h(\sigma)x(t-\sigma)d\sigma$$

where  $x(t)$  is the input,  $y(t)$  is the output, and  $h(t)$  the impulse response of the system.

A nonlinear system without memory can be described with a Taylor series as:

$$y(t) = \sum_{n=1}^{\infty} a_n [x(t)]^n$$

where  $x(t)$  is the input and  $y(t)$  is the output. The  $a_n$  are Taylor series coefficients.

# Volterra Series

A Volterra series combines the above two representations to describe a nonlinear system with memory

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} du_1 \dots \int_{-\infty}^{\infty} du_n g_n(u_1, \dots, u_n) \prod_{r=1}^n x(t - u_r)$$

$$\begin{aligned} y(t) = & \frac{1}{1!} \int_{-\infty}^{\infty} du_1 g_1(u_1) x(t - u_1) \leftarrow \text{impulse response} \\ & + \frac{1}{2!} \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 g_2(u_1, u_2) x(t - u_1) x(t - u_2) \leftarrow \text{higher-order impulse responses} \\ & + \frac{1}{2!} \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 \int_{-\infty}^{\infty} du_3 g_3(u_1, u_2, u_3) x(t - u_1) x(t - u_2) g_2(u_1, u_2) x(t - u_1) x(t - u_2) x(t - u_3) \\ & + \dots \end{aligned}$$

where  $x(t)$  is the input and  $y(t)$  is the output and the  $g_n(u_1, \dots, u_n)$  are called the *Volterra kernels*