ECE 546
Lecture 03
Waveguides

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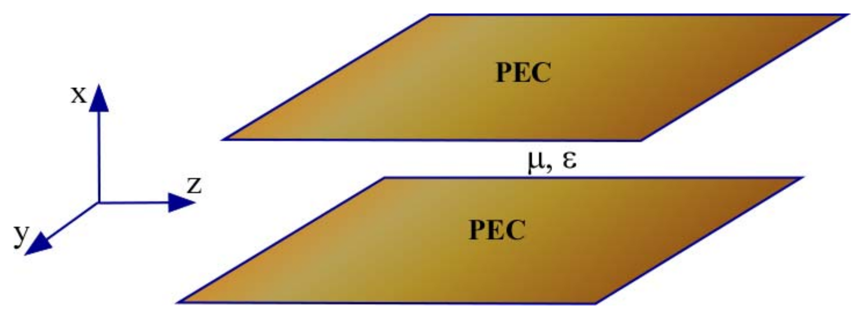
Jose E. Schutt-Aine
Electrical & Computer Engineering
University of Illinois
jesa@illinois.edu

Parallel-Plate Waveguide

Maxwell's Equations ➔ $\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = \mathbf{0}$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2\mu\varepsilon E_x$$



$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2\mu\varepsilon E_y$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2\mu\varepsilon E_z$$

TE Modes

For a parallel-plate waveguide, the plates are infinite in the *y*-extent; we need to study the propagation in the *z*-direction. The following assumptions are made in the wave equation

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

$$\Rightarrow \text{Assume } E_y \text{ only}$$

These two conditions define the **TE modes** and the wave equation is simplified to read

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu\varepsilon E_y \qquad (¥)$$

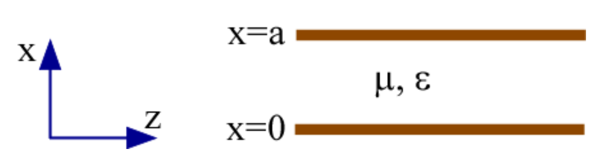
Phasor Solution

General solution (forward traveling wave)

$$E_y(x,z) = e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x} \right]$$

At $x = 0$, $E_y = 0$ which leads to $A + B = 0$. Therefore, $A = -B = E_o/2j$, where E_o is an arbitrary constant

$$E_y(x,z) = E_o e^{-j\beta_z z} \sin\beta_x x$$



a is the distance separating the two PEC plates

Dispersion Relation

At $x = a$, $E_y(x, z) = 0$ ➔ $E_o e^{-j\beta_z z} \sin \beta_x a = 0$

This leads to: $\beta_x a = m\pi$, where $m = 1, 2, 3, \ldots$

$$\beta_x = \frac{m\pi}{a}$$

Moreover, from the differential equation (¥), we get the *dispersion relation*

$$\beta_z^2 + \beta_x^2 = \omega^2 \mu\varepsilon = \beta^2$$

which leads to $\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$

Guidance Condition

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

where m = 1, 2, 3 ... Since propagation is to take place in the z direction, for the wave to propagate, we must have $\beta_z^{\,2} > 0$, or

$$\omega^2 \mu\varepsilon > \left(\frac{m\pi}{a}\right)^2$$

This leads to the following *guidance condition* which will insure wave propagation

$$f > \frac{m}{2a\sqrt{\mu\varepsilon}}$$

Cutoff Frequency

The *cutoff frequency* f_c is defined to be at the onset of propagation

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} \qquad\qquad \lambda_c = \frac{v}{f_c} = \frac{2a}{m}$$

Each mode is referred to as the TE_m mode. It is obvious that there is no TE_0 mode and the first TE mode is the TE_1 mode.

The *cutoff frequency* is the frequency below which the mode associated with the index ***m*** will not propagate in the waveguide. Different modes will have different cutoff frequencies.

Magnetic Field for TE Modes

From $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$

we have $\mathbf{H} = \dfrac{-1}{j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$

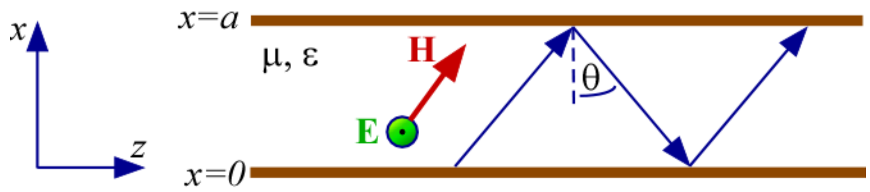
which leads to

$$H_x = -\frac{\beta_z}{\omega\mu} E_o e^{-j\beta_z z} \sin \beta_x x$$

$$H_z = +\frac{j\beta_x}{\omega\mu} E_o e^{-j\beta_z z} \cos \beta_x x$$

The magnetic field for TE modes has 2 components

E & H Fields for TE Modes



As can be seen, there is no H_y component, therefore, the TE solution has E_y, H_x and H_z only.

From the dispersion relation, it can be shown that the propagation vector components satisfy the relations
$\beta_z = \beta \sin\theta$, $\beta_x = \beta \cos\theta$ where θ is the angle of incidence of the propagation vector with the normal to the conductor plates.

Phase and Group Velocities

The phase and group velocities are given by

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{c}{\sqrt{1 - \frac{f_c^2}{f^2}}} \qquad \text{and} \qquad v_g = \frac{\partial \omega}{\partial \beta_z} = c\sqrt{1 - \frac{f_c^2}{f^2}}$$

The effective guide impedance is given by:

$$\eta_{TE} = \frac{E_y}{-H_x} = \frac{\eta_o}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

Transverse Magnetic (TM) Modes

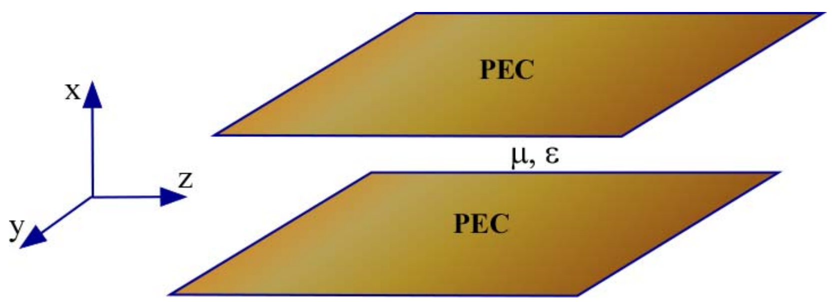
The magnetic field also satisfies the wave equation:

Maxwell's Equations ➔ $\nabla^2\mathbf{H} + \omega^2\mu\varepsilon\mathbf{H}=\mathbf{0}$

$$\frac{\partial^2 H_x}{\partial x^2}+\frac{\partial^2 H_x}{\partial y^2}+\frac{\partial^2 H_x}{\partial z^2}=-\omega^2\mu\varepsilon H_x$$

$$\frac{\partial^2 H_y}{\partial x^2}+\frac{\partial^2 H_y}{\partial y^2}+\frac{\partial^2 H_y}{\partial z^2}=-\omega^2\mu\varepsilon H_y$$

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+\frac{\partial^2 H_z}{\partial z^2}=-\omega^2\mu\varepsilon H_z$$



TM Modes

For TM modes, we assume

$$\Rightarrow \frac{\partial}{\partial y} = 0, \text{ but } \frac{\partial}{\partial x} \neq 0 \text{ and } \frac{\partial}{\partial z} \neq 0$$

➔Assume H_y only

These two conditions define the *TM modes* and the equations are simplified to read

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu \varepsilon H_y$$

<u>General solution (forward traveling wave)</u>

$$H_y(x,z) = e^{-j\beta_z z}\left[Ae^{-j\beta_x x} + Be^{+j\beta_x x} \right]$$

Electric Field for TM Modes

From $\nabla \times \mathbf{H} = -j\omega\varepsilon\mathbf{E}$

we get $$\mathbf{E} = \frac{1}{j\omega\varepsilon}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix}$$

This leads to

$$E_x(x,z) = \frac{\beta_z}{\omega\varepsilon} e^{-j\beta_z z}\left[A e^{-j\beta_x x} + B e^{+j\beta_x x} \right]$$

$$E_z(x,z) = \frac{\beta_x}{\omega\varepsilon} e^{-j\beta_z z}\left[-A e^{-j\beta_x x} + B e^{+j\beta_x x} \right]$$

TM Modes Fields

At $x=0$, $E_z = 0$ which leads to $A = B = H_o/2$ where H_o is an arbitrary constant. This leads to

$$H_y(x,z) = H_o e^{-j\beta_z z}\cos\beta_x x$$

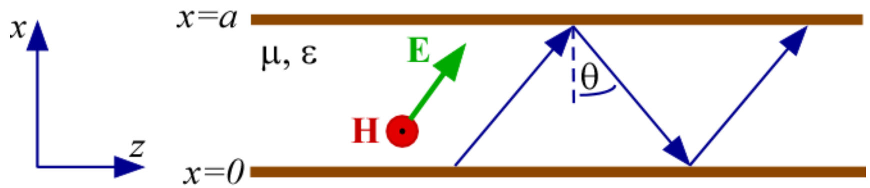
$$E_x(x,z) = \frac{\beta_z}{\omega\varepsilon} H_o e^{-j\beta_z z}\cos\beta_x x$$

$$E_z(x,z) = \frac{j\beta_x}{\omega\varepsilon} H_o e^{-j\beta_z z}\sin\beta_x x$$

At $x = a$, $E_z = 0$ which leads to

$\beta_x a = m\pi$, where $m = 0, 1, 2, 3, \ldots$

E & H Fields for TM Modes



$$\beta_x = \frac{m\pi}{a}$$

This defines the TM modes which have only H_y, E_x and E_z components.

The effective guide impedance is given by:

$$\eta_{TM} = \frac{E_x}{H_y} = \eta_o \sqrt{1 - \frac{f_c^2}{f^2}}$$

The electric field for TM modes has 2 components

E & H Fields for TM Modes

THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A PARALLEL-PLATE WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

This defines the **TM modes**; each mode is referred to as the TM_m mode. It can be seen from that *m=0* is a valid choice; it is called the TM_0, or *transverse electromagnetic* or TEM mode. For this mode and,

TEM Mode

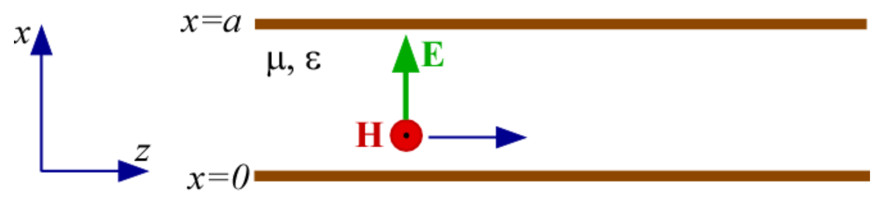
β_x=0 and $\beta_z = \beta$. There are no x variations of the fields within the waveguide. The TEM mode has a cutoff frequency at DC and is always present in the waveguide.

$$H_y = H_o e^{-j\beta_z z}$$

$$E_x = \frac{\beta_z}{\omega\varepsilon} H_o e^{-j\beta_z z} = \sqrt{\frac{\mu}{\varepsilon}} H_o e^{-j\beta_z z}$$

$$E_z = 0$$

The propagation characteristics of the TEM mode do not vary with frequency



The TEM mode is the *fundamental* mode on a parallel-plate waveguide

Power for TE Modes

Time-Average Poynting Vector $\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\}$

<u>TE modes</u>

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{\hat{\mathbf{y}}E_y \times \left[\hat{\mathbf{x}}H_x^* + \hat{\mathbf{z}}H_z^*\right]\right\}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{\hat{\mathbf{z}}\frac{|E_o|^2}{\omega\mu}\beta_z \sin^2\beta_x x + \hat{\mathbf{x}}j\frac{|E_o|^2}{\omega\mu}\beta_x \cos\beta_x x \sin\beta_x x\right\}$$

$$\langle \mathbf{P} \rangle = \hat{\mathbf{z}}\frac{|E_o|^2}{2\omega\mu}\beta_z \sin^2\beta_x x$$

Power for TM Modes

TM modes

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{\left[\hat{\mathbf{x}}E_x + \hat{\mathbf{z}}E_z\right]\times \hat{\mathbf{y}}H_y^*\right\}$$

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}\left\{\hat{\mathbf{z}}\frac{\left|H_o\right|^2}{\omega\varepsilon}\beta_z \cos^2\beta_x x - \hat{\mathbf{x}}j\frac{\left|H_o\right|^2}{\omega\varepsilon}\beta_x \sin\beta_x x \cos\beta_x x\right\}$$

$$\langle \mathbf{P} \rangle = \hat{\mathbf{z}}\frac{\left|H_o\right|^2}{2\omega\varepsilon}\beta_z \cos^2\beta_x x$$

The total time-average power is found by integrating <**P**> over the area of interest.