ECE 546
Lecture - 05
Transmission Lines

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Maxwell's Equations

$$\nabla \times E = -\frac{\partial B}{\partial t}$$ **Faraday's Law of Induction**

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$ **Ampère's Law**

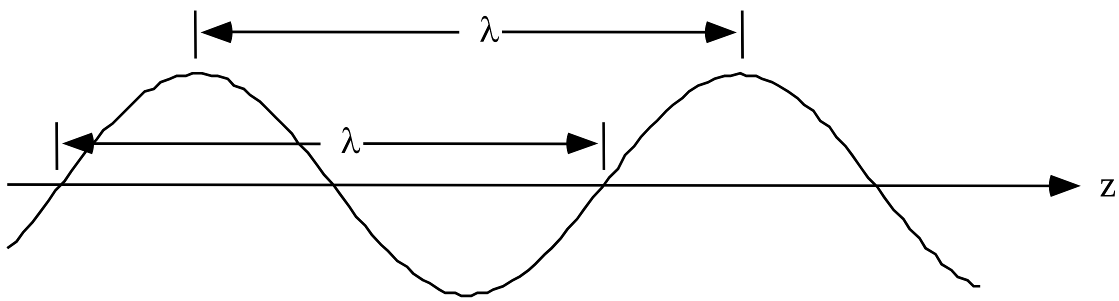
$$\nabla \cdot D = \rho$$ **Gauss' Law for electric field**

$$\nabla \cdot B = 0$$ **Gauss' Law for magnetic field**

Constitutive Relations

$$B = \mu H \qquad D = \varepsilon E$$

Why Transmission Line?



Wavelength : λ

$$\lambda = \frac{\text{propagation velocity}}{\text{frequency}}$$

Why Transmission Line?

In Free Space

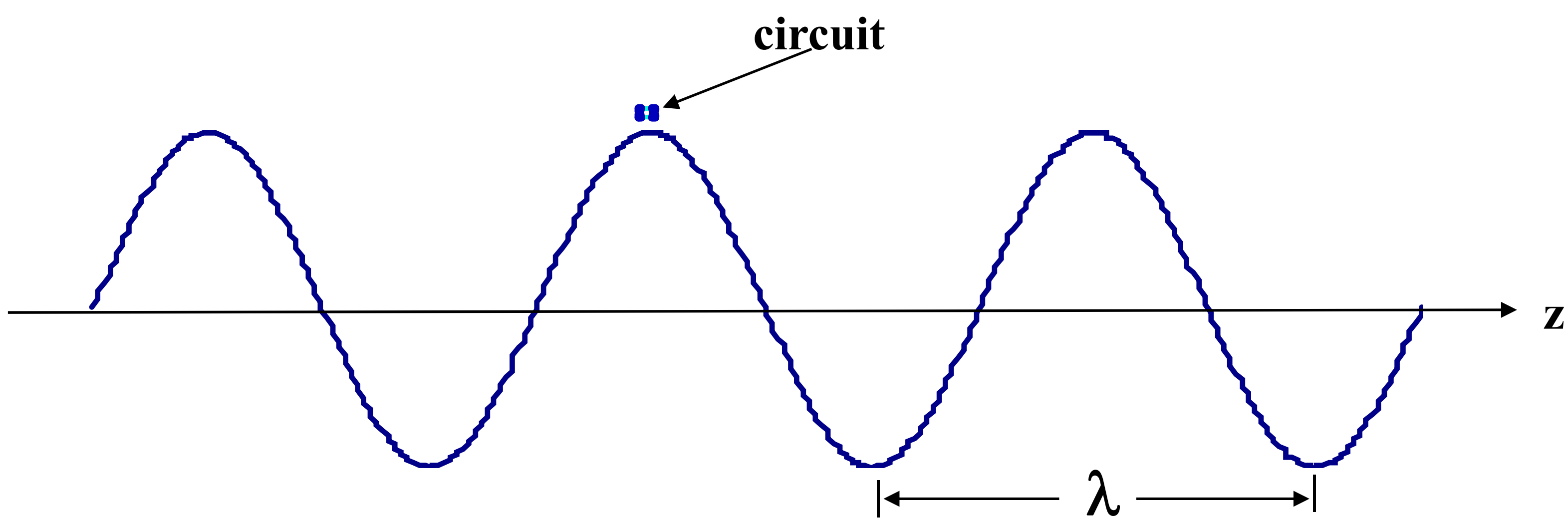
At 10 KHz : $\lambda = 30$ km

At 10 GHz : $\lambda = 3$ cm

Transmission line behavior is prevalent when the structural dimensions of the circuits are comparable to the wavelength.

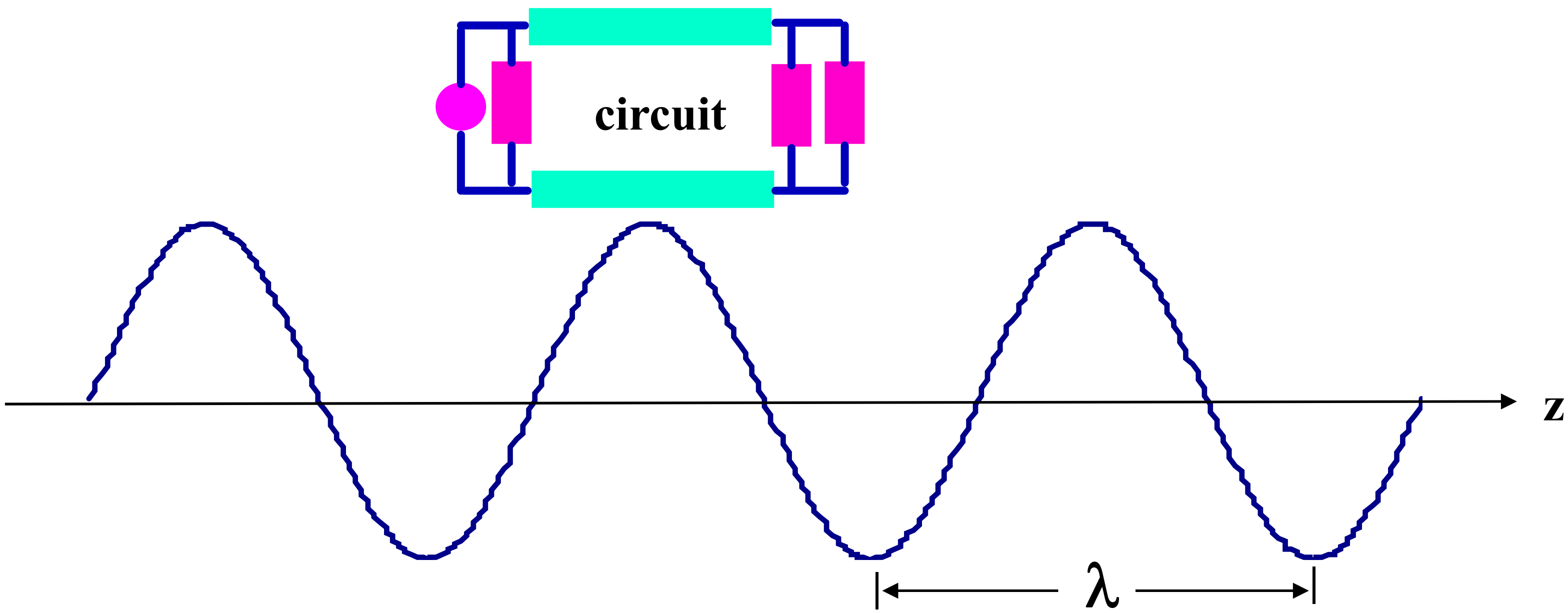
Justification for Transmission Line

Let d be the largest dimension of a circuit



If d << λ, a lumped model for the circuit can be used

Justification for Transmission Line



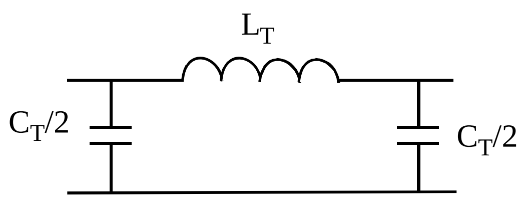
If d ≈ λ, or d > λ then use transmission line model

Modeling Interconnections

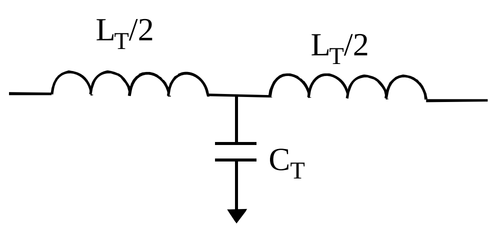
Low Frequency

Short

Mid-range Frequency

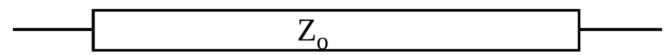


or



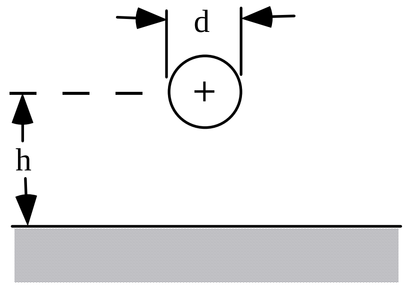
Lumped Reactive CKT

High Frequency



Transmission Line

Single wire near ground



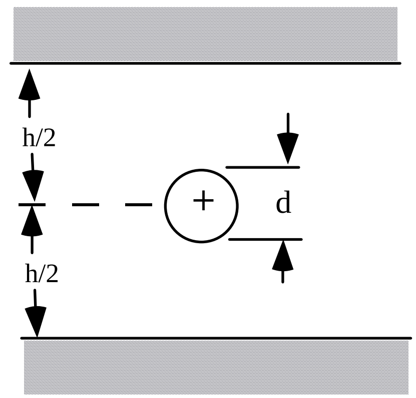
$$\text{for } d \ll h, \quad Z_o = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{4h}{d}\right)$$

$$Z_o = 120 \cosh^{-1}(D/d)$$

$$C = \frac{2\pi\varepsilon}{ln\left(\frac{4h}{d}\right)}$$

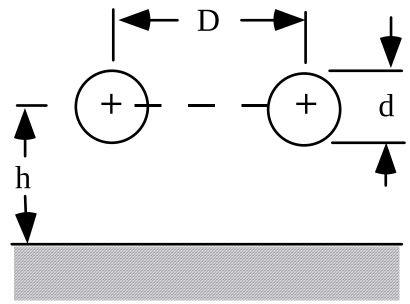
$$L = \frac{\mu}{2\pi}\ln\frac{4h}{d}$$

Single wire between grounded parallel planes ground return



$$Z_o = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{4h}{\pi d}\right)$$

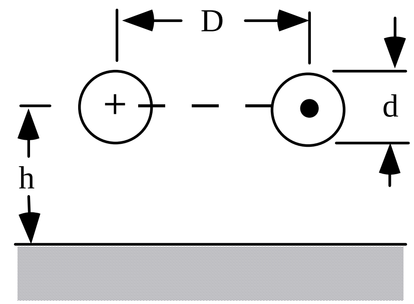
Wires in parallel near ground



for d << D, h

$$Z_O = \left(69/\sqrt{\varepsilon_r}\right)\log_{10}\left\{\left(4h/d\right)\sqrt{1+\left(2h/D\right)^2}\right\}$$

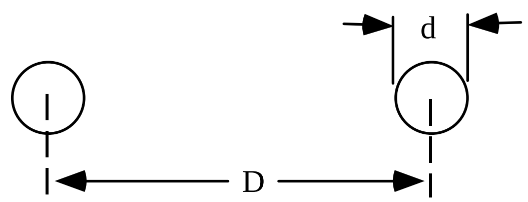
Balanced, near ground



for d << D, h

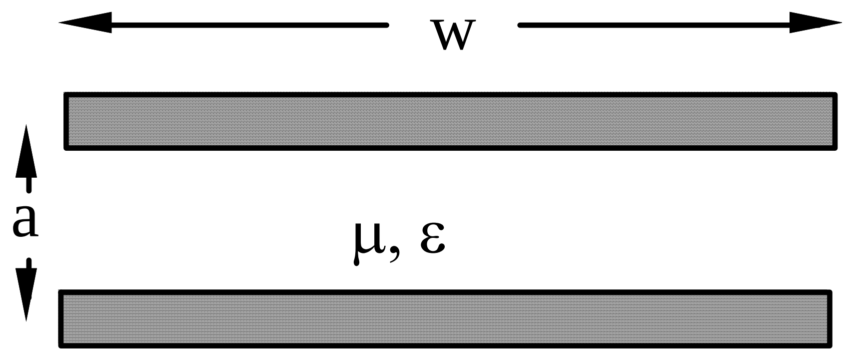
$$Z_O = \left(276/\sqrt{\varepsilon_r}\right)\log_{10}\left\{\frac{(2D/d)}{\sqrt{1+(D/2h)^2}}\right\}$$

Open 2-wire line in air



$$Z_o = 120 \cosh^{-1}(D/d)$$

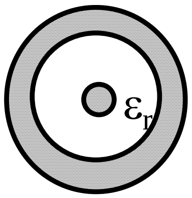
Parallel-plate Transmission Line



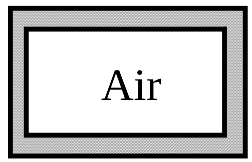
$$L = \frac{\mu a}{w}$$

$$C = \frac{\varepsilon w}{a}$$

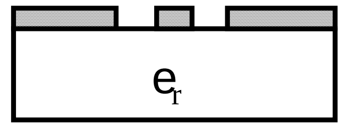
Types of Transmission Lines



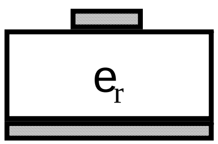
Coaxial line



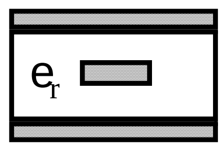
Waveguide



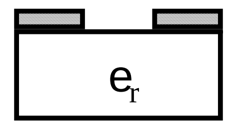
Coplanar line



Microstrip

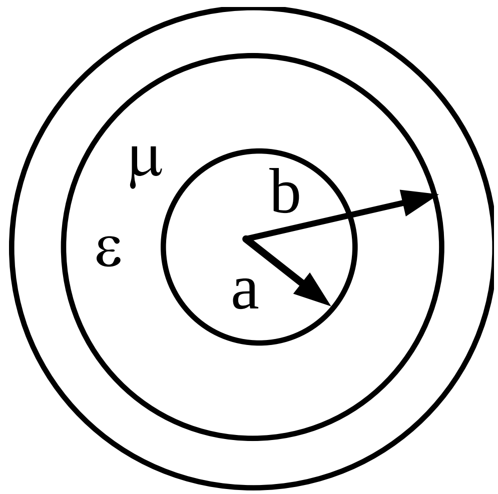


Stripline



Slot line

Coaxial Transmission Line

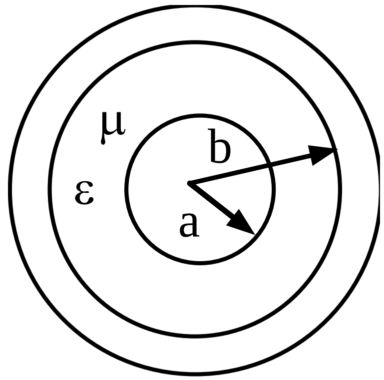


TEM Mode of Propagation

$$L = \mu \ln \frac{b}{a}$$

$$C = \frac{2\pi\varepsilon}{\ln(b/a)}$$

Coaxial Air Lines



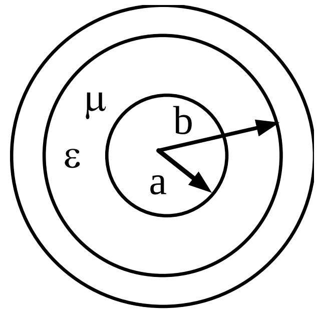
Infinite Conductivity

$$Z_o = \frac{\sqrt{\mu/\varepsilon}}{2\pi}\ln\left(b/a\right)$$

Finite Conductivity

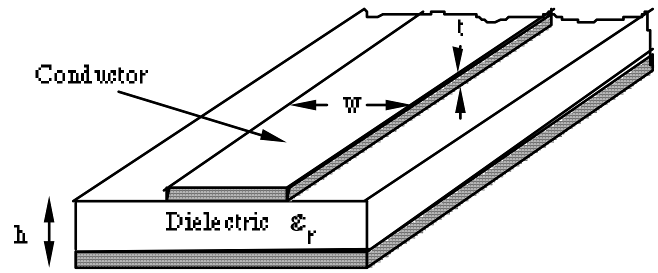
$$Z_o = \frac{\sqrt{\mu/\varepsilon}}{2\pi}\ln\left(b/a\right)\left[1+\frac{\left(1/a+1/b\right)}{4\sqrt{\pi f \mu\sigma}\ln(b/a)}\left(1-j\right)\right]$$

Coaxial Connector Standards

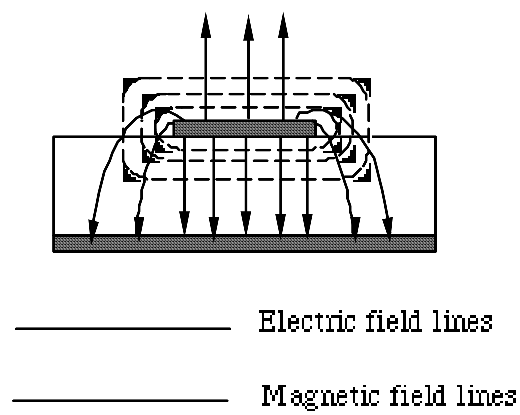


Connector	Frequency Range
14 mm	DC - 8.5 GHz
GPC-7	DC - 18 GHz
Type NDC - 18 GHz	
3.5 mm	DC - 33 GHz
2.92 mm	DC - 40 GHz
2.4 mm	DC - 50 GHz
1.85 mm	DC - 65 GHz
1.0 mm	DC - 110 GHz

Microstrip

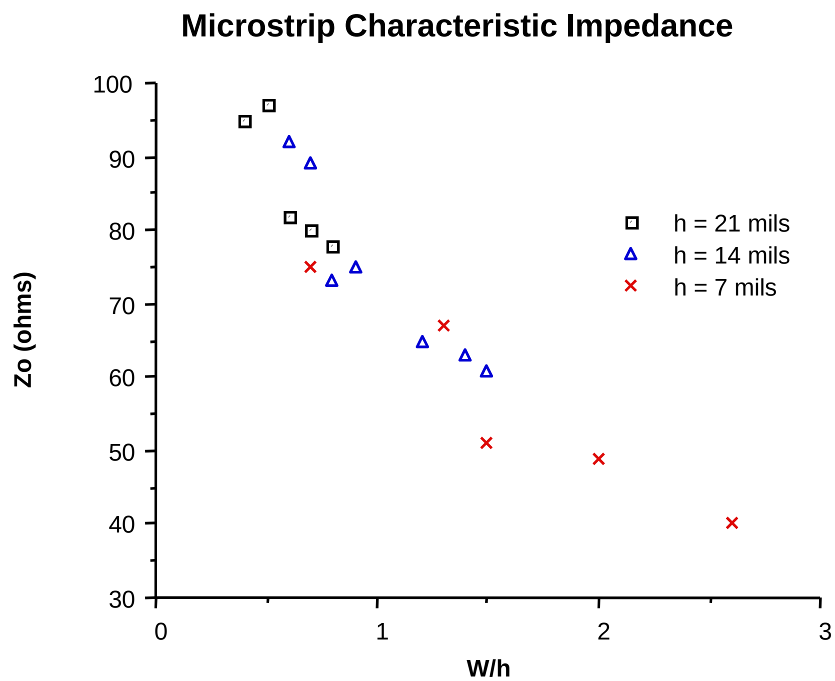


(a)



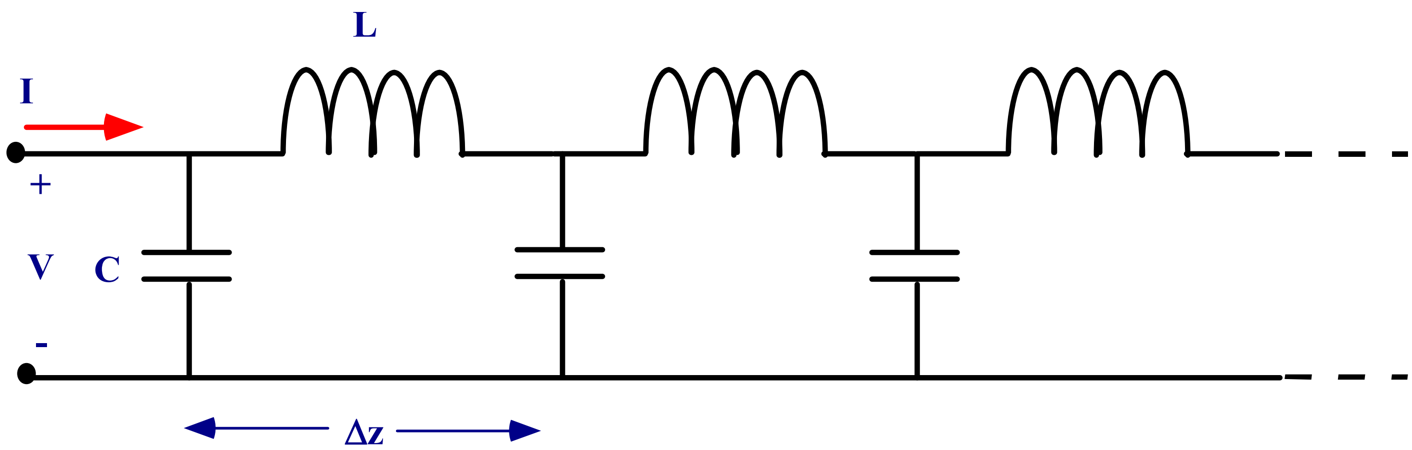
(b)

Microstrip



dielectric constant : 4.3.

Telegraphers' Equations



***L*: Inductance per unit length.**

***C*: Capacitance per unit length.**

$$-\frac{\partial V}{\partial z} = L\frac{\partial I}{\partial t}$$

$$-\frac{\partial I}{\partial z} = C\frac{\partial V}{\partial t}$$

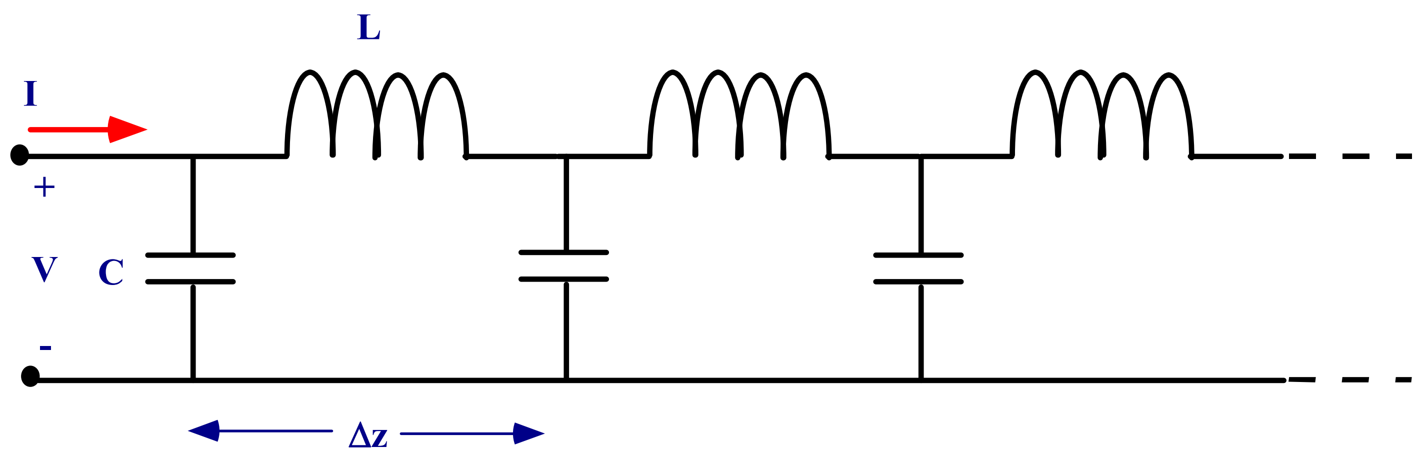
Assume time-harmonic dependence

$$V,\ I \sim e^{j\omega t}$$

$$-\frac{\partial V}{\partial z} = j\omega L I$$

$$-\frac{\partial I}{\partial z} = j\omega C V$$

TL Solutions

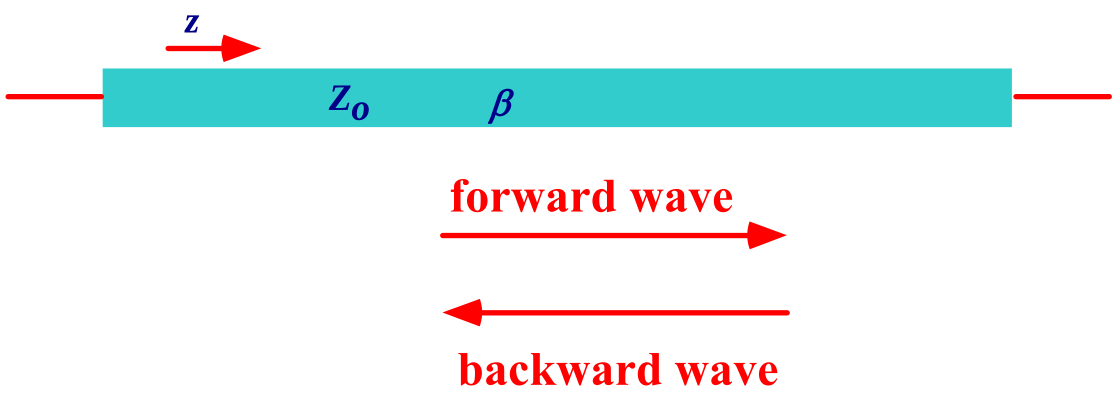


$$-\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) = -\frac{\partial^2 V}{\partial z} = j\omega L \frac{\partial I}{\partial z} = -j\omega L j\omega C V \quad \Longrightarrow \quad \frac{\partial^2 V}{\partial z^2} = -\omega^2 LCV$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial I}{\partial z}\right) = -\frac{\partial^2 I}{\partial z} = j\omega C \frac{\partial V}{\partial z} = -j\omega L j\omega C I \quad \Longrightarrow \quad \frac{\partial^2 I}{\partial z^2} = -\omega^2 CLI$$

TL Solutions

(Frequency Domain)



$$\beta = \omega\sqrt{LC}$$

$$V(z) = \overbrace{V_+ e^{-j\beta z}}^{Forward\ Wave} + \overbrace{V_- e^{+j\beta z}}^{Backward\ Wave}$$

$$Z_o = \sqrt{\frac{L}{C}}$$

$$I(z) = \underbrace{\frac{V_+}{Z_o} e^{-j\beta z}}_{Forward\ Wave} - \underbrace{\frac{V_-}{Z_o} e^{+j\beta z}}_{Backward\ Wave}$$

TL Solutions

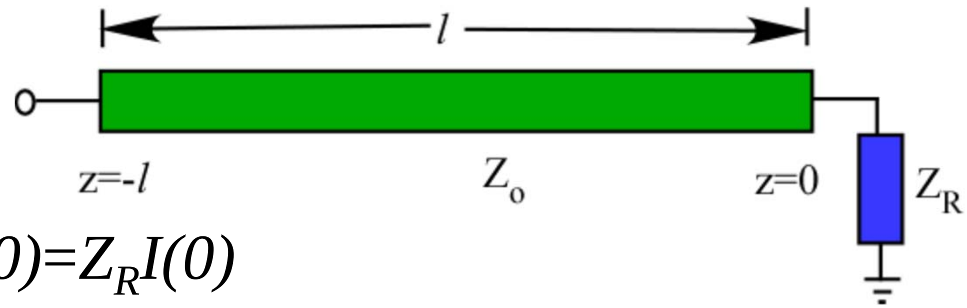
Propagation constant $\beta = \omega\sqrt{LC}$ **Propagation velocity** $v = \dfrac{1}{\sqrt{LC}}$

Characteristic impedance $Z_o = \sqrt{\dfrac{L}{C}}$ **Wavelength** $\lambda = \dfrac{v}{f}$

$$V(z,t) = \overbrace{V_+ \cos(\omega t - \beta z)}^{Forward\ Wave} + \overbrace{V_- \cos(\omega t + \beta z)}^{Backward\ Wave}$$

$$I(z,t) = \underbrace{\frac{V_+}{Z_o}\cos(\omega t - \beta z)}_{Forward\ Wave} - \underbrace{\frac{V_-}{Z_o}\cos(\omega t + \beta z)}_{Backward\ Wave}$$

Reflection Coefficient



At $z=0$, we have $V(0)=Z_R I(0)$

But from the TL equations:

$$V(0) = V_+ + V_-$$

$$I(0) = \frac{V_+}{Z_o} - \frac{V_-}{Z_o}$$

$$\frac{Z_R}{Z_o}\left(V_+ - V_-\right) = V_+ + V_-$$

Which gives $V_- = \Gamma_R V_+$

where $\Gamma_R = \dfrac{Z_R - Z_o}{Z_R + Z_o}$ is the load reflection coefficient

Reflection Coefficient

- If $Z_R = Z_o$, Γ_R=0, no reflection, the line is matched

- If $Z_R = 0$, short circuit at the load, Γ_R=-1

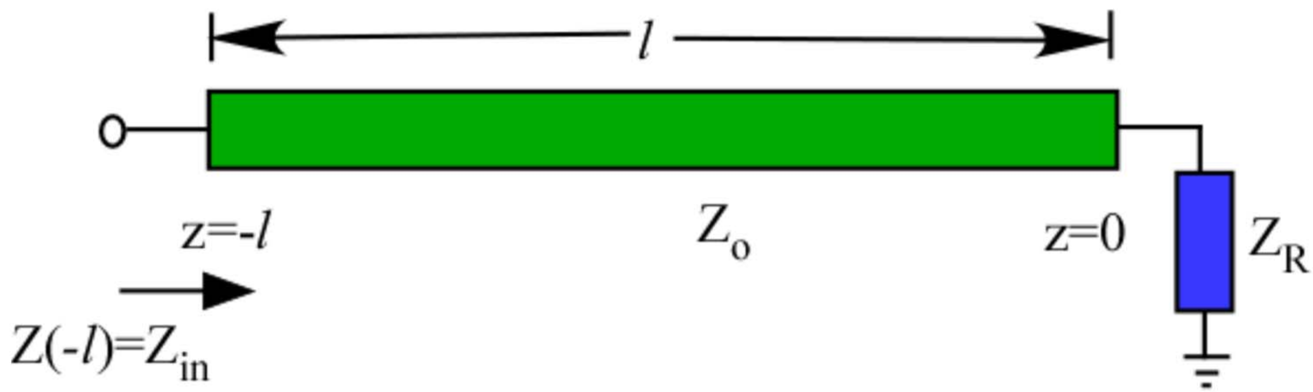
- If $Z_R \rightarrow$ inf, open circuit at the load, Γ_R=+1

V and *I* can be written in terms of Γ_R

$$V(z) = V_+\left[e^{-j\beta z} + \Gamma_R e^{+j\beta z}\right] \qquad V(z) = V_+ e^{-j\beta z}\left[1 + \Gamma_R e^{+2j\beta z}\right]$$

$$I(z) = \frac{V_+}{Z_o}\left[e^{-j\beta z} - \Gamma_R e^{+j\beta z}\right] \qquad I(z) = \frac{V_+ e^{-j\beta z}}{Z_o}\left[1 - \Gamma_R e^{+2j\beta z}\right]$$

Generalized Impedance

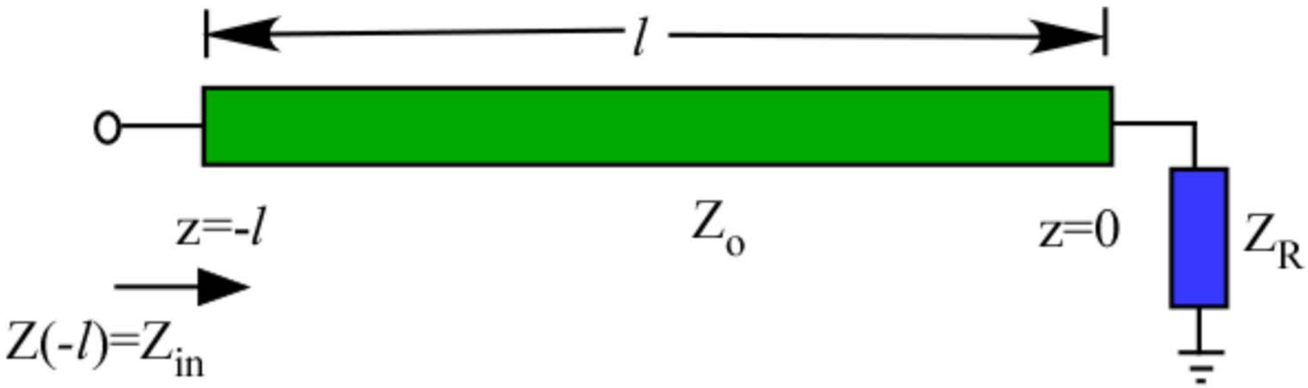


$$Z(z)=\frac{V(z)}{I(z)}=Z_o\left[\frac{e^{-j\beta z}+\Gamma_R e^{+j\beta z}}{e^{-j\beta z}-\Gamma_R e^{+j\beta z}}\right]$$

$$Z(-l)=Z_o\left[\frac{Z_R+jZ_o\tan\beta l}{Z_o+jZ_R\tan\beta l}\right]$$

Impedance transformation equation

Generalized Impedance



- **Short circuit Z_R=0, line appears inductive for 0 < l < λ/2**

$$Z(-l) = jZ_o \tan \beta l$$

- **Open circuit Z_R ➔ inf, line appears capacitive for 0 < l < λ/2**

$$Z(-l) = \frac{Z_o}{j \tan \beta l}$$

- **If l = λ/4, the line is a quarter-wave transformer**

$$Z(-l) = \frac{Z_o^2}{Z_R}$$

Generalized Reflection Coefficient

$$\Gamma(z) = \frac{\text{Backward traveling wave at } z}{\text{Forward traveling wave at } z} = \frac{V_b(z)}{V_f(z)}$$

$$\Gamma(z) = \frac{V_- e^{+j\beta z}}{V_+ e^{-j\beta z}} = \frac{V_-}{V_+} e^{+2j\beta z} = \Gamma_R e^{+2j\beta z}$$

Reflection coefficient transformation equation ⇨ $\Gamma(-l) = \Gamma_R e^{-2j\beta l}$

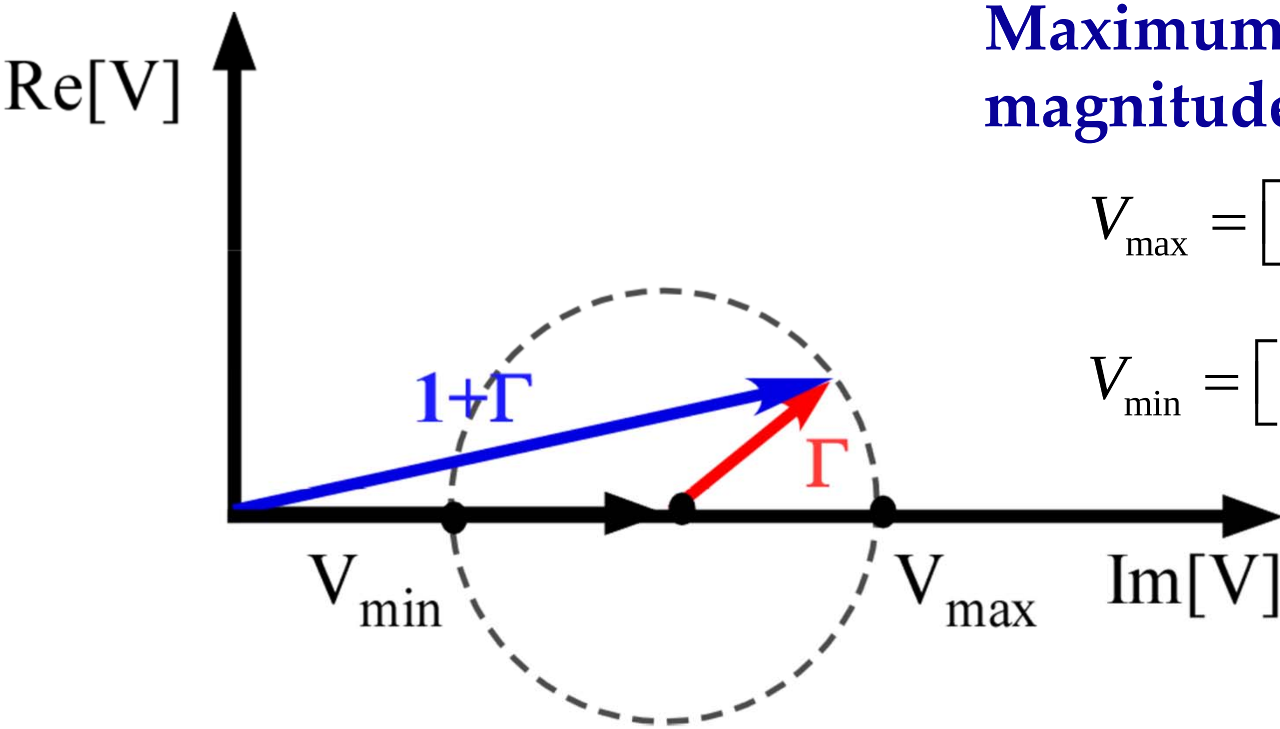
$$Z(z) = Z_o \frac{1+\Gamma(z)}{1-\Gamma(z)} \qquad \Gamma(z) = \frac{Z(z)-Z_o}{Z(z)+Z_o}$$

Voltage Standing Wave Ratio (VSWR)

$$V(z) = V_+ e^{-j\beta z}\left[1 + \Gamma_R e^{+2j\beta z}\right]$$

We follow the magnitude of the voltage along the TL

$$\left|V(z)\right| = \left|V_+ e^{-j\beta z}\right|\left|1 + \Gamma_R e^{+2j\beta z}\right| = \left|V_+\right|\left|1 + \Gamma_R e^{+2j\beta z}\right|$$



Maximum and minimum magnitudes given by

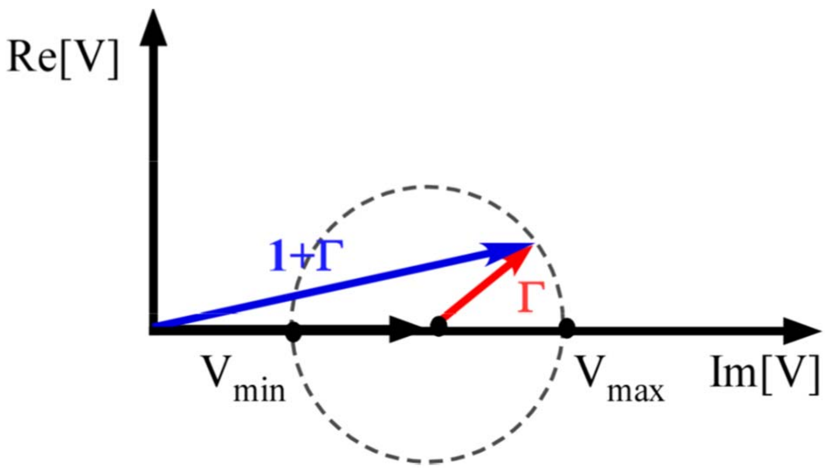
$$V_{\max} = \left[1 + \left|\Gamma_R\right|\right]$$

$$V_{\min} = \left[1 - \left|\Gamma_R\right|\right]$$

Voltage Standing Wave Ratio (VSWR)

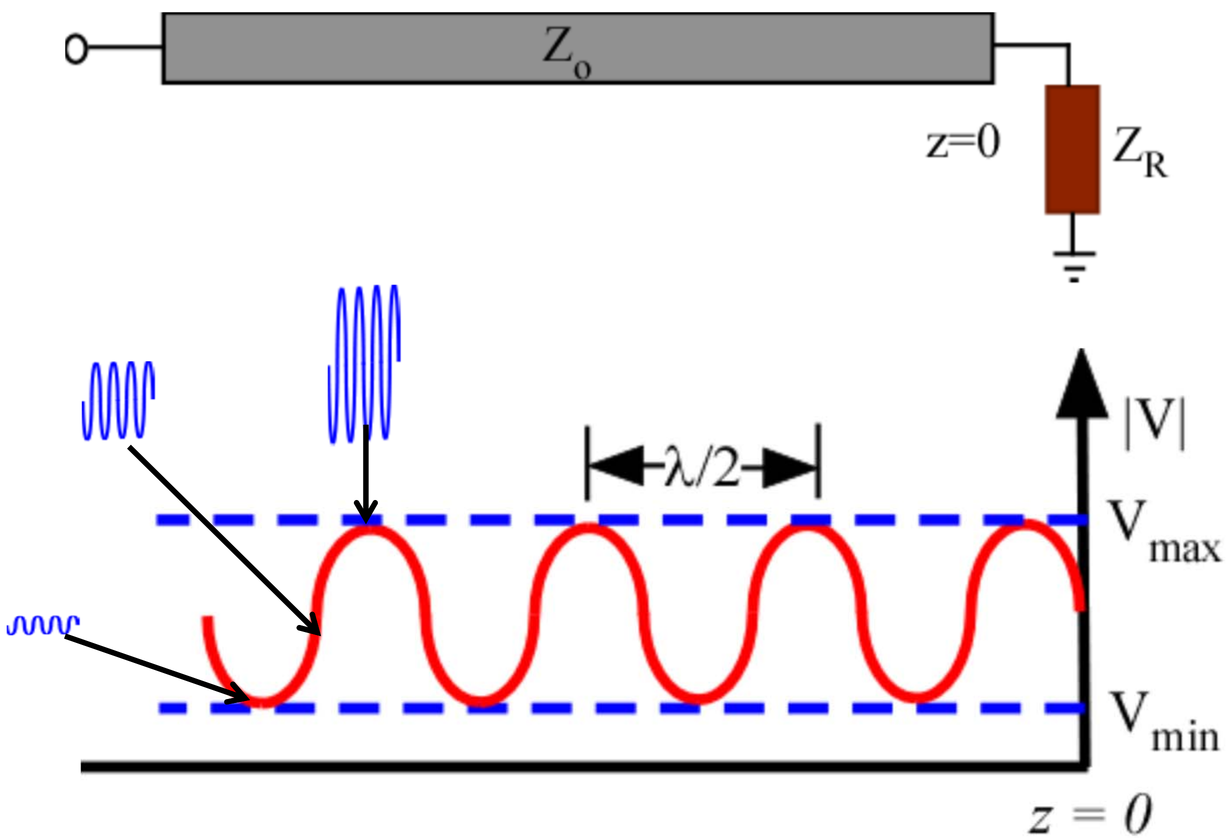
Define Voltage Standing Wave Ratio as:

$$VSWR = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma_R|}{1-|\Gamma_R|}$$



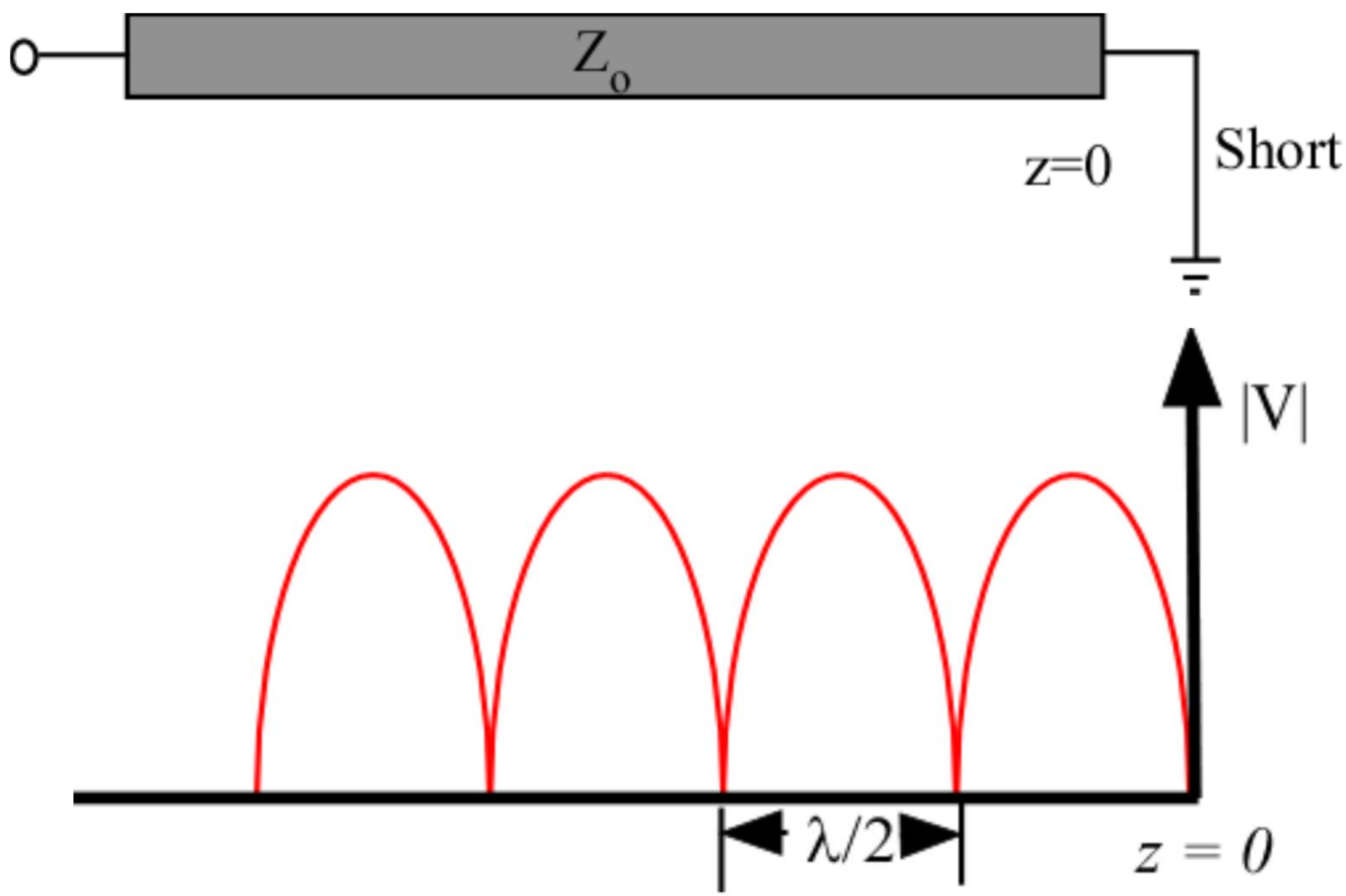
It is a measure of the interaction between forward and backward waves

VSWR – Arbitrary Load



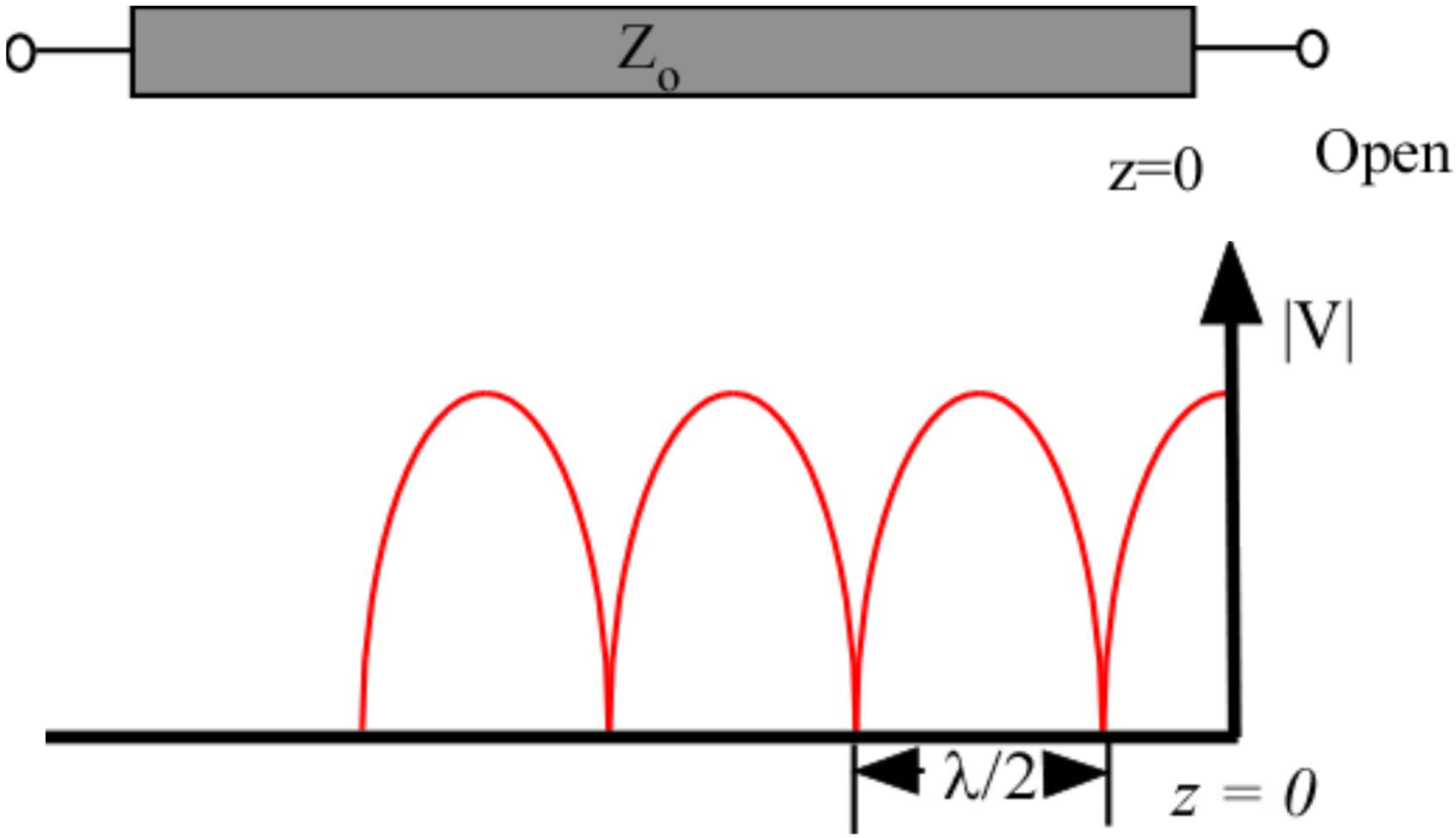
Shows variation of amplitude along line

VSWR – For Short Circuit Load



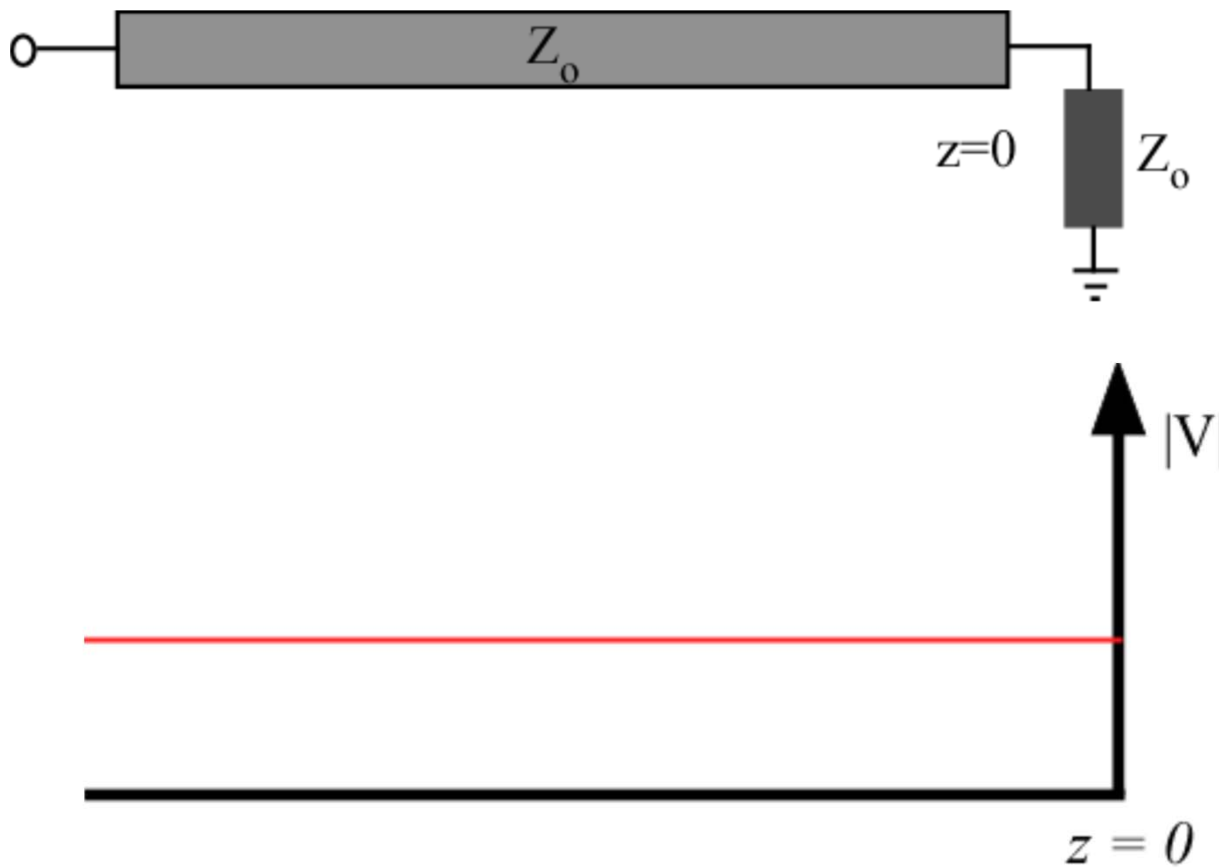
Voltage minimum is reached at load

VSWR – For Open Circuit Load



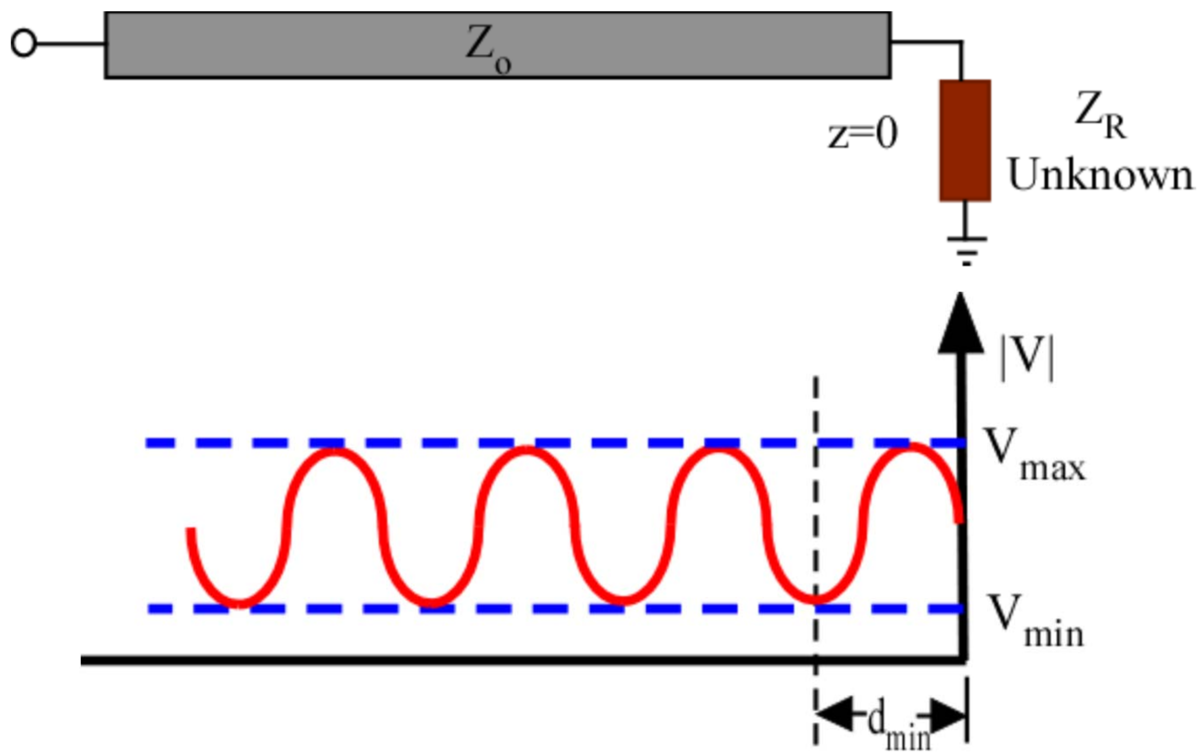
Voltage maximum is reached at load

VSWR – For Open Matched Load



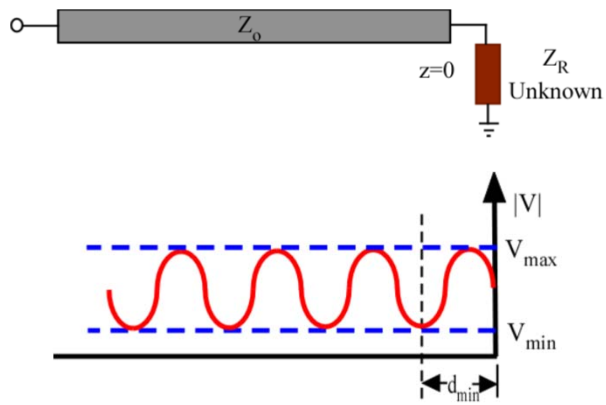
No variation in amplitude along line

Application: Slotted-Line Measurement



- Measure VSWR = V_{max}/V_{min}
- Measure location of first minimum

Application: Slotted-Line Measurement



At minimum, $\Gamma(z) = \text{pure real} = -\left|\Gamma_R\right|$

Therefore, $\Gamma(-d_{\min}) = \Gamma_R e^{-2j\beta d_{\min}} = -\left|\Gamma_R\right|$

So, $\Gamma_R = -\left|\Gamma_R\right| e^{+2j\beta d_{\min}}$

Since $\left|\Gamma_R\right| = \dfrac{VSWR-1}{VSWR+1}$ **then** $\Gamma_R = -\left(\dfrac{VSWR-1}{VSWR+1}\right)e^{+2j\beta d_{\min}}$

and $Z_R = Z_o\left(\dfrac{1+\Gamma_R}{1-\Gamma_R}\right)$

Summary of TL Equations

Voltage

$$V(z) = V_+ e^{-j\beta z}\left[1+\Gamma_R e^{+2j\beta z}\right]$$

Current

$$I(z) = \frac{V_+}{Z_o} e^{-j\beta z}\left[1-\Gamma_R e^{+2j\beta z}\right]$$

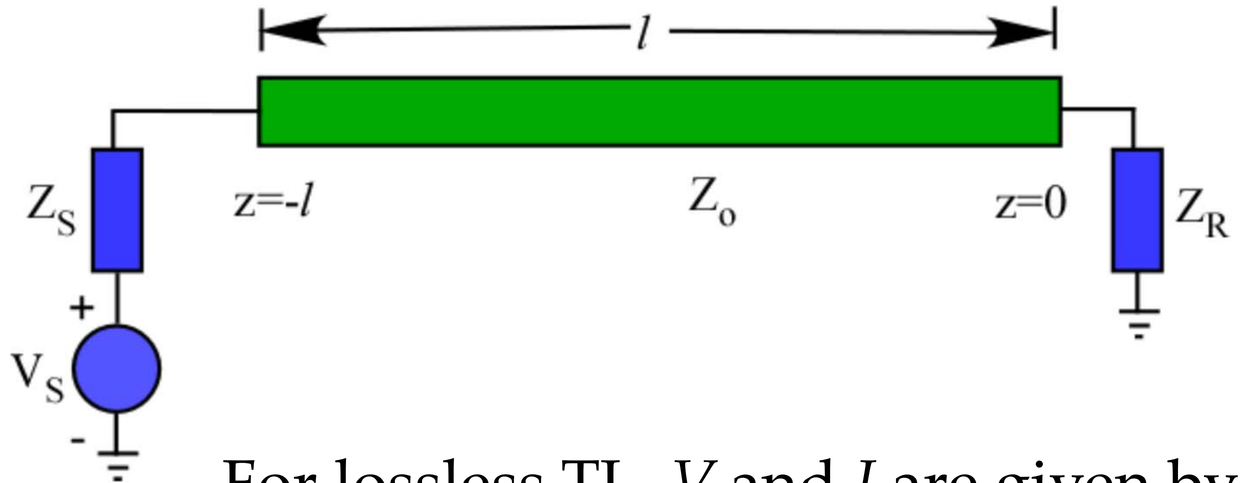
Impedance Transformation ➔ $$Z(-l) = Z_o\left[\frac{Z_R + jZ_o \tan\beta l}{Z_o + jZ_R \tan\beta l}\right]$$

Reflection Coefficient Transformation ➔ $$\Gamma(-l) = \Gamma_R e^{-2j\beta l}$$

Reflection Coefficient – to Impedance ➔ $$Z(z) = Z_o \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

Impedance to Reflection Coefficient ➔ $$\Gamma(z) = \frac{Z(z)-Z_o}{Z(z)+Z_o}$$

Determining V_+



For lossless TL, V and I are given by

$$V(z) = V_+ e^{-j\beta z}\left[1 + \Gamma_R e^{+2j\beta z}\right]$$

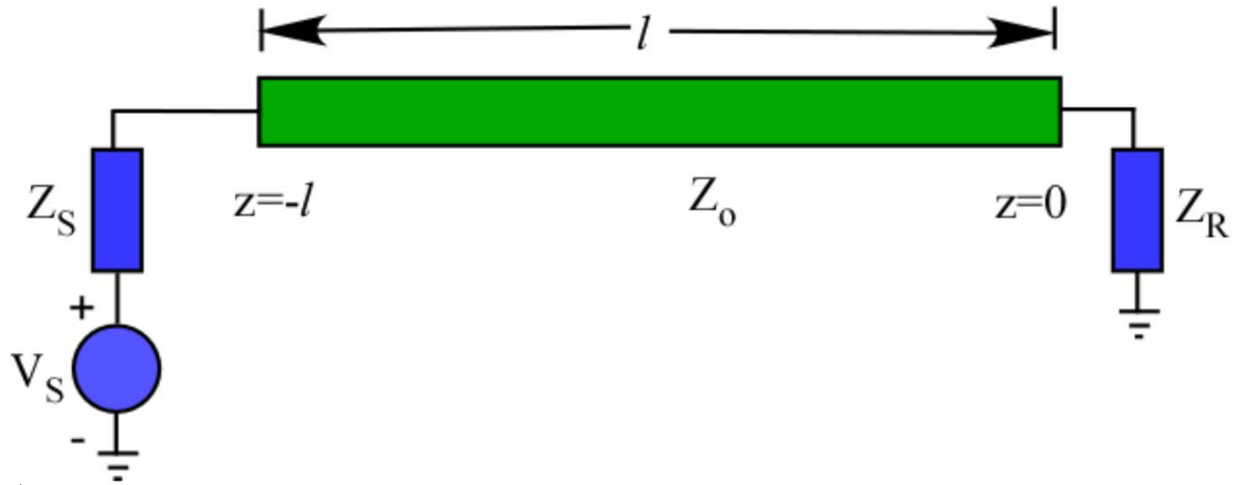
reflection coefficient at the load

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o}$$

$$I(z) = \frac{V_+ e^{-j\beta z}}{Z_o}\left[1 - \Gamma_R e^{+2j\beta z}\right]$$

$$At\ z = -l, \quad V_S = Z_S I(-l) + V(-l)$$

Determining V_+



this leads to

$$V_S = V_+ e^{+j\beta l}\left(1+\Gamma_R e^{-2j\beta l}\right)+\frac{Z_S}{Z_o}V_+ e^{+j\beta l}\left(1-\Gamma_R e^{-2j\beta l}\right)$$

or

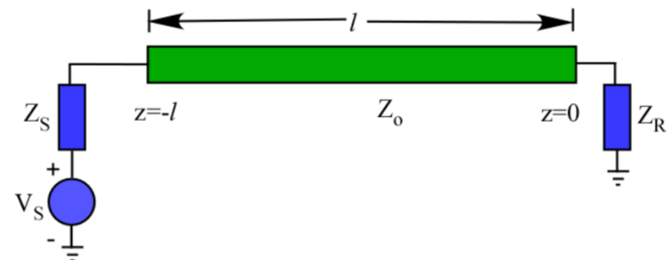
$$V_S = V_+\left(e^{+j\beta l}+\Gamma_R e^{-j\beta l}+\frac{Z_S}{Z_o}e^{+j\beta l}-\Gamma_R\frac{Z_S}{Z_o}e^{-j\beta l}\right)$$

$$V_S = V_+\left(e^{+j\beta l}\left(1+\frac{Z_S}{Z_o}\right)+\Gamma_R e^{-j\beta l}\left(1-\frac{Z_S}{Z_o}\right)\right)$$

Determining V_+

Divide through by $\left(1+\frac{Z_S}{Z_o}\right)=\frac{1}{T_S}$

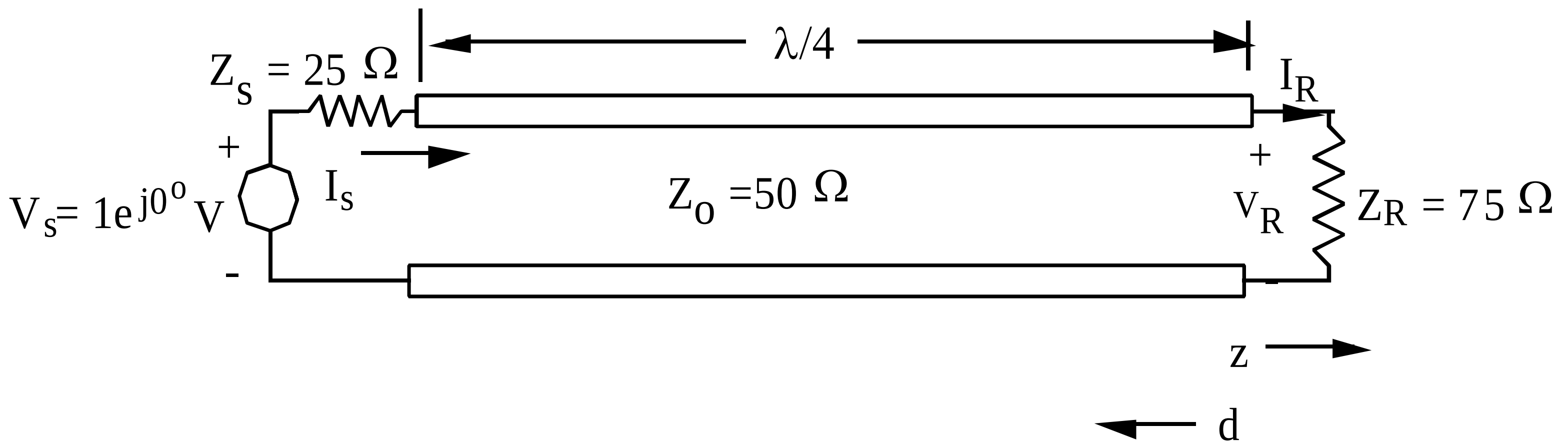
$$V_+\left(e^{+j\beta l}-\Gamma_S\Gamma_R e^{-j\beta l}\right)=T_S V_S$$



with $T_S=\left(1+\frac{Z_S}{Z_o}\right)^{-1}=\frac{Z_o}{Z_S+Z_o}$ and $\Gamma_S=\frac{Z_S-Z_o}{Z_S+Z_o}$

From which $$V_+=\frac{T_S V_S e^{-j\beta l}}{1-\Gamma_S\Gamma_R e^{-2j\beta l}}$$

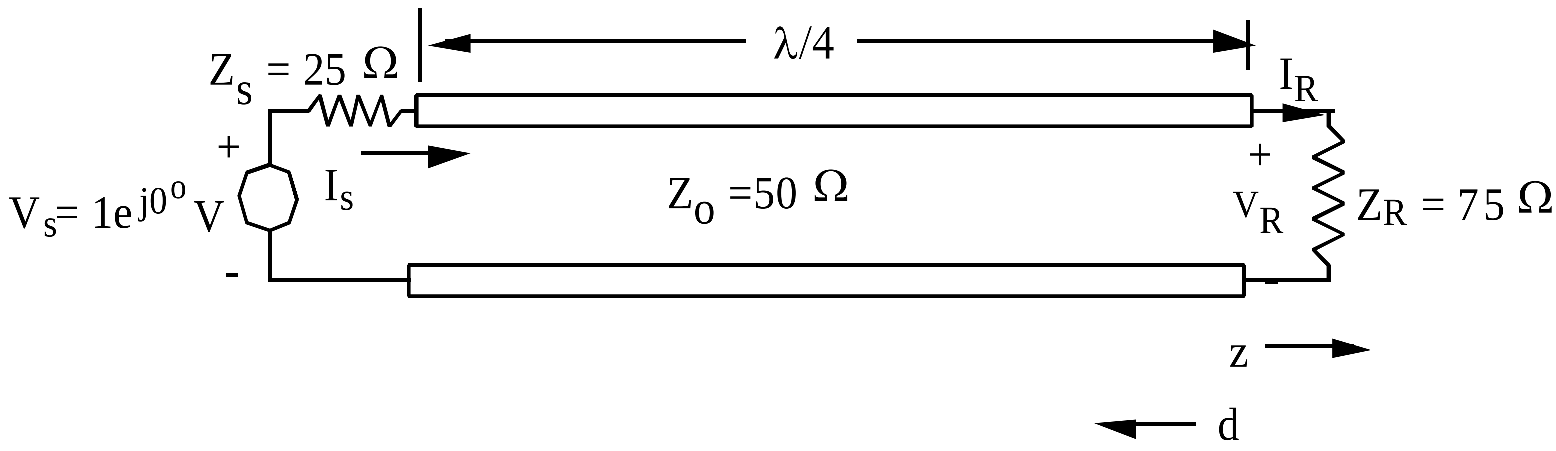
TL Example



A signal generator having an internal resistance $Z_s = 25\ \Omega$ and an open circuit phasor voltage $V_s = 1e^{j0}$ volt is connected to a 50-Ω lossless transmission line as shown in the above picture. The load impedance is $Z_R = 75\ \Omega$ and the line length is $\lambda/4$.

Find the magnitude and phase of the load current I_R.

TL Example – Cont'



$$V_+ = \frac{T_s V_s e^{-j\beta l}}{1-\Gamma_R \Gamma_s e^{-2j\beta l}} \qquad T_s = \frac{Z_o}{Z_s+Z_o} = \frac{50}{50+25} = 2/3$$

$$\Gamma_s = \frac{Z_s - Z_o}{Z_s+Z_o} = \frac{25-50}{25+50} = -1/3 \qquad \Gamma_R = \frac{Z_R - Z_o}{Z_R+Z_o} = \frac{75-50}{75+50} = 1/5$$

TL Example – Cont'

$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2} \Rightarrow e^{-j\beta l} = -j$$

$$V_+ = \frac{(2/3)(1)(-j)}{1-(-1/3)(1/5)(-1)} = \frac{-j2/3}{1-1/15} = -j5/7$$

$$V_+ = -j0.714285 \text{ V}$$

$$\mathrm{I_R} = \frac{V_+}{Z_o}\left[1-\Gamma_R\right] = -j\frac{0.714285}{50}\left[1-0.2\right] = -j\frac{0.714285\times 0.8}{50}$$

$$\mathrm{I_R} = -j0.014285 \text{ A}$$

Geometric Series Expansion

Since $\left|\Gamma_S \Gamma_R e^{-2j\beta l}\right| \leq 1$

V_+ can be expanded in a geometric series form

$$V_+ = \frac{T_S V_S e^{-j\beta l}}{1 - \Gamma_S \Gamma_R e^{-2j\beta l}} \qquad V_+ = T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^k e^{-2j\beta kl} e^{-j\beta l}$$

$$V(z) = T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^k e^{-j\beta l(2k+1)} e^{-j\beta z} + T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^{k+1} e^{-j\beta l(2k+1)} e^{+j\beta z}$$

$$\beta = \frac{\omega}{v}$$

$$V(z) = T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^k e^{-j\frac{\omega}{v}\left[z+(2k+1)l\right]} + T_S \sum_{k=0}^{\infty} V_S \Gamma_S^k \Gamma_R^{k+1} e^{-j\frac{\omega}{v}\left[z+(2k+1)l\right]}$$

TL Time-Domain Solution

$$v(z,t) = T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^k v_s \left(t - \frac{(2k+1)l + z}{v_o} \right)$$

$$+ T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^{k+1} v_s \left(t - \frac{(2k+1)l - z}{v_o} \right)$$

at z=0

$$v(0,t) = T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^k v_s \left(t - \frac{(2k+1)l}{v_o} \right)$$

$$+ T_S \sum_{k=0}^{\infty} \Gamma_S^k \Gamma_R^{k+1} v_s \left(t - \frac{(2k+1)l}{v_o} \right)$$

TL Time-Domain Solution

At z=-l

$$v(-l,t)=T_S\sum_{k=0}^{\infty}\Gamma_S^k\Gamma_R^k v_s\left(t-\frac{(2k+1)l+l}{v_o}\right)$$

$$+T_S\sum_{k=0}^{\infty}\Gamma_S^k\Gamma_R^{k+1} v_s\left(t-\frac{(2k+1)l+l}{v_o}\right)$$

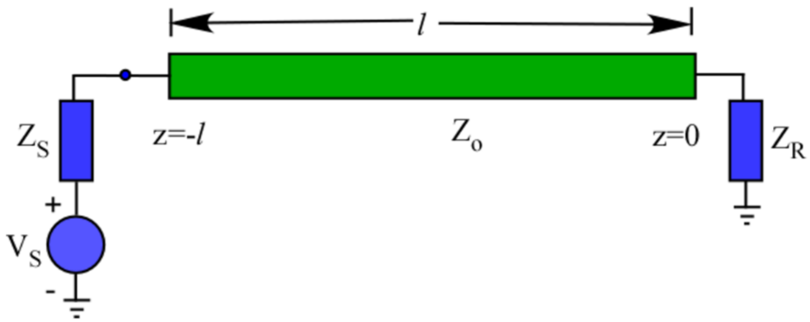
$$v(-l,t)=T_S\sum_{k=0}^{\infty}\Gamma_S^k\Gamma_R^k v_s\left(t-\frac{2kl}{v_o}\right)$$

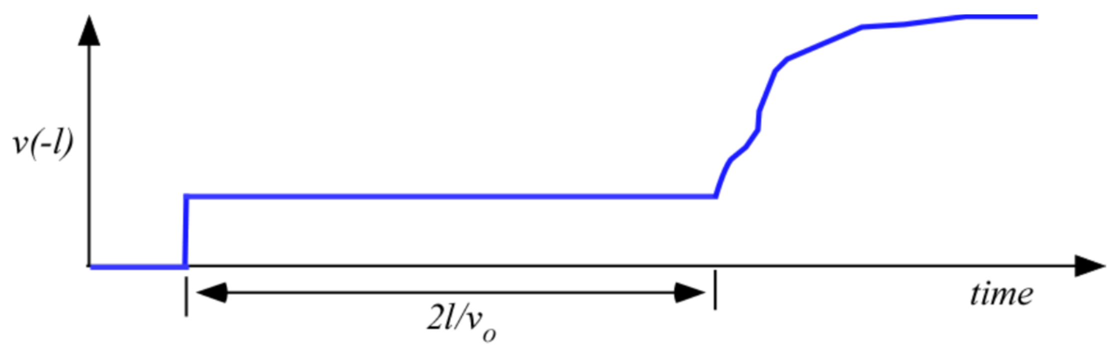
$$+T_S\sum_{k=0}^{\infty}\Gamma_S^k\Gamma_R^{k+1} v_s\left(t-\frac{2(k+1)l}{v_o}\right)$$

TL - Time-Domain Reflectometer

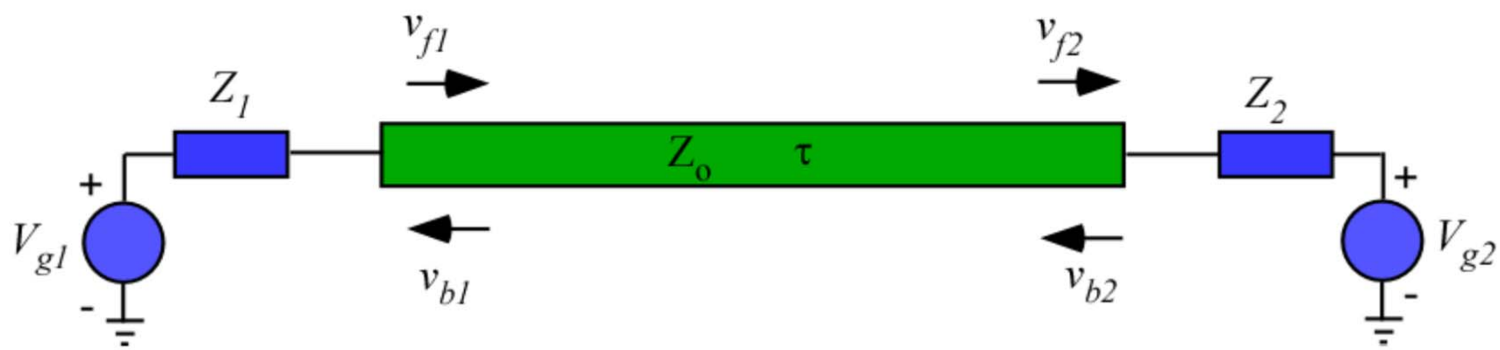
For TDR, $Z_S = Z_o \rightarrow \Gamma_S = 0$, and retain only $k=1$

$$v(-l,t) = T_S v_s(t) + T_S \Gamma_R v_s\left(t - \frac{2l}{v_o}\right)$$





Wave Shifting Method



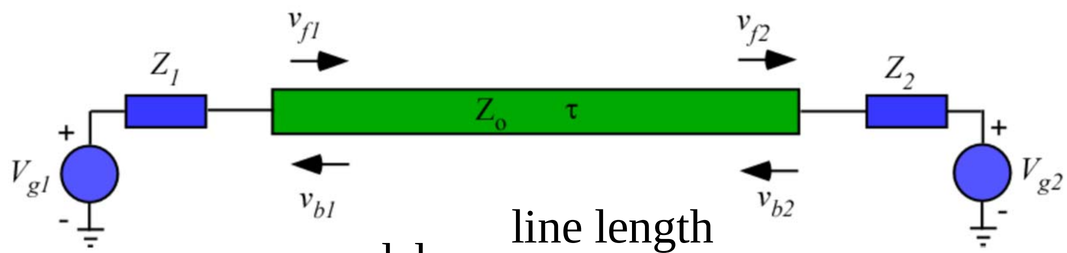
$v_{f1}(t)$ Forward traveling wave at port 1 (measured at near end of line)

$v_{f2}(t)$ Forward traveling wave at port 2 (measured at far end of line)

$v_{b1}(t)$ Backward traveling wave at port 1 (measured at near end of line)

$v_{b2}(t)$ Backward traveling wave at port 2 (measured at far end of line)

Wave Shifting Solution*



$$\tau = \text{delay} = \frac{\text{line length}}{\text{velocity}}$$

$$v_{f1}(t) = \Gamma_1 v_{b2}(t-\tau) + T_1 V_{g1}(t)$$
$$v_{b2}(t) = \Gamma_2 v_{f1}(t-\tau) + T_2 V_{g2}(t)$$

$$v_{f2}(t) = v_{f1}(t-\tau)$$
$$v_{b1}(t) = v_{b2}(t-\tau)$$

$$v_1(t) = v_{f1}(t) + v_{b1}(t)$$
$$v_2(t) = v_{f2}(t) + v_{b2}(t)$$

$$\Gamma_1 = \frac{Z_1 - Z_o}{Z_1 + Z_o}$$

$$T_1 = \frac{Z_o}{Z_1 + Z_o}$$

$$\Gamma_2 = \frac{Z_2 - Z_o}{Z_2 + Z_o}$$

$$T_2 = \frac{Z_o}{Z_2 + Z_o}$$

***Schutt-Aine & Mittra, *Trans. Microwave Theory Tech.*, pp. 529-536, vol. 36 March 1988.**

Frequency Dependence of Lumped Circuit Models

- At higher frequencies, a lumped circuit model is no longer accurate for interconnects and one must use a distributed model

- Transition frequency depends on the dimensions and relative magnitude of the interconnect parameters.

$$f \approx \frac{0.3 \times 10^9}{10d\sqrt{\varepsilon_r}} \qquad t_r \approx \frac{0.35}{f}$$

Lumped Circuit or Transmission Line?

- **Determine frequency or bandwidth of signal**
  - RF/Microwave: f= operating frequency
  - Digital: $f=0.35/t_r$

- **Determine the propagation velocity and wavelength**
  - Material medium $v=c/(\varepsilon_r)^{1/2}$
  - Obtain wavelength $\lambda=v/f$

- **Compare wavelenth with feature size**
  - If $\lambda >> d$, use lumped circuit: $L_{tot}= L^*$ length, $C_{tot}= C^*$ length
  - If $\lambda \approx 10d$ or $\lambda<10d$, use transmission-line model

Frequency Dependence of Lumped Circuit Models

Level	Dimension	Frequency	Edge rate
PCB line	10 in	> 55 MHz	< 7ns
Package	1 in	> 400 MHz	< 0.9 ns
VLSI int*	100 um	> 8 GHz	< 50 ps

* Using RC criterion for distributed effect