

ECE 546

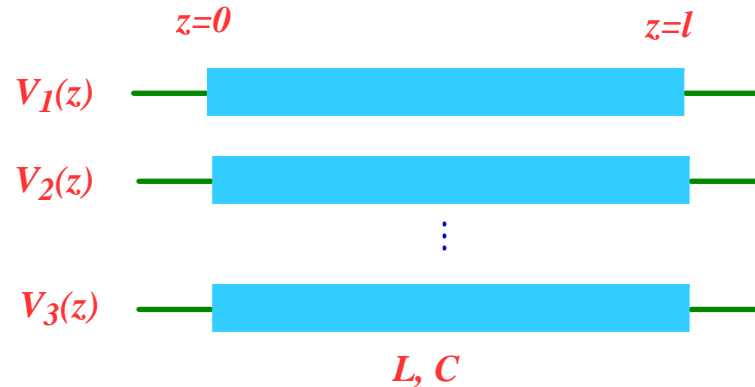
Lecture - 07

Multiconductors

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TELGRAPHER'S EQUATION FOR N COUPLED TRANSMISSION LINES



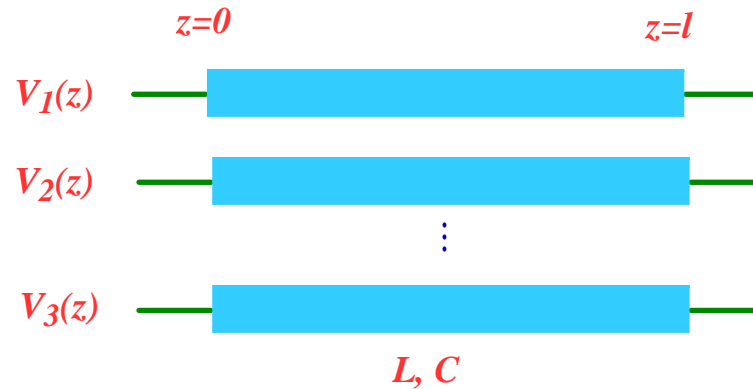
$$-\frac{\partial V}{\partial z} = L \frac{\partial I}{\partial t}$$

$$-\frac{\partial I}{\partial z} = C \frac{\partial V}{\partial t}$$

V and I are the line voltage and line current VECTORS respectively (dimension n).

N-LINE SYSTEM

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ V_n \end{bmatrix}$$



$$I = \begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ I_n \end{bmatrix}$$

L and C are the inductance and capacitance MATRICES respectively

$$L = \begin{bmatrix} L_{11} & L_{12} & \cdot & \cdot \\ L_{21} & L_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & L_{nn} \end{bmatrix}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot \\ C_{21} & C_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & C_{nn} \end{bmatrix}$$

N-LINE ANALYSIS

$$\frac{\partial^2 V}{\partial z^2} = LC \frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial z^2} = CL \frac{\partial^2 I}{\partial t^2}$$

In general $LC \neq CL$ and LC and CL are not symmetric matrices.

GOAL: Diagonalize LC and CL which will result in a transformation on the variables V and I .

Diagonalize LC and CL is equivalent to finding the eigenvalues of LC and CL .

Since LC and CL are adjoint, they must have the same eigenvalues.

MODAL ANALYSIS

$$\frac{\partial^2 EV}{\partial z^2} = ELCE^{-1} \frac{\partial^2 EV}{\partial t^2}$$

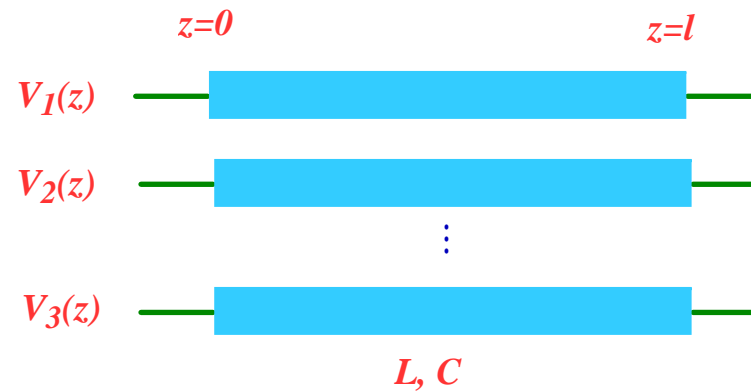
$$\frac{\partial^2 HI}{\partial z^2} = HCLH^{-1} \frac{\partial^2 HI}{\partial t^2}$$

***LC* and *CL* are adjoint matrices. Find matrices *E* and *H* such that**

$$ELCE^{-1} = HCLH^{-1} = \Lambda_m^2$$

is the diagonal eigenvalue matrix whose entries are the inverses of the modal velocities squared.

MODAL ANALYSIS



$$\frac{\partial^2 I_m}{\partial z^2} = \Lambda_m^2 \frac{\partial^2 I_m}{\partial t^2}$$

$$\frac{\partial^2 V_m}{\partial z^2} = \Lambda_m^2 \frac{\partial^2 V_m}{\partial t^2}$$

$$V_m = EV$$

$$I_m = HI$$

Second-order differential equation in modal space

EIGENVECTORS

$$V_m = EV$$

$$I_m = HI$$

V_m and I_m are the modal voltage and current vectors respectively.

$$V_m = \begin{bmatrix} V_{m1} \\ V_{m2} \\ \cdot \\ \cdot \\ V_{mn} \end{bmatrix}$$

$$I_m = \begin{bmatrix} I_{m1} \\ I_{m2} \\ \cdot \\ \cdot \\ I_{mn} \end{bmatrix}$$

EIGEN ANALYSIS

- * **A scalar λ is an eigenvalue of a matrix A if there exists a vector X such that $AX = \lambda X$. (i.e. for which multiplication by a matrix is equivalent to an elongation of the vector).**
- * **The vector X which satisfies the above requirement is an eigenvector of A .**
- * **The eigenvalues λ of A satisfy the relation $|A - \lambda I| = 0$ where I is the unit matrix.**

EIGEN ANALYSIS

Assume A is an $n \times n$ matrix with n distinct eigenvalues, then,

- * A has n linearly independent eigenvectors.
- * The n eigenvectors can be arranged into an $n \times n$ matrix E ; the eigenvector matrix.
- * Finding the eigenvalues of A is equivalent to diagonalizing A .
- * Diagonalization is achieved by finding the eigenvector matrix E such that EAE^{-1} is a diagonal matrix.

EIGEN ANALYSIS

For an n-line system, it can be shown that

- * LC can be transformed into a diagonal matrix whose entries are the eigenvalues.**
- * LC possesses n distinct eigenvalues (possibly degenerate).**
- * There exist n eigenvectors which are linearly independent.**
- * Each eigenvalue is associated with a mode; the propagation velocity of that mode is the inverse of the eigenvalue.**

EIGEN ANALYSIS

- * **Each eigenvector is associated to an eigenvalue and therefore to a particular mode.**
- * **Each normalized eigenvector represents the relative line excitation required to excite the associated mode.**

EIGENVECTORS

E : Voltage eigenvector matrix

$$ELCE^{-1} = \Lambda_m^2$$

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

H : Current eigenvector matrix

$$HCLH^{-1} = \Lambda_m^2$$

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Eigenvalues and Eigenvectors

$$ELCE^{-1} = \Lambda_m^2$$

gives

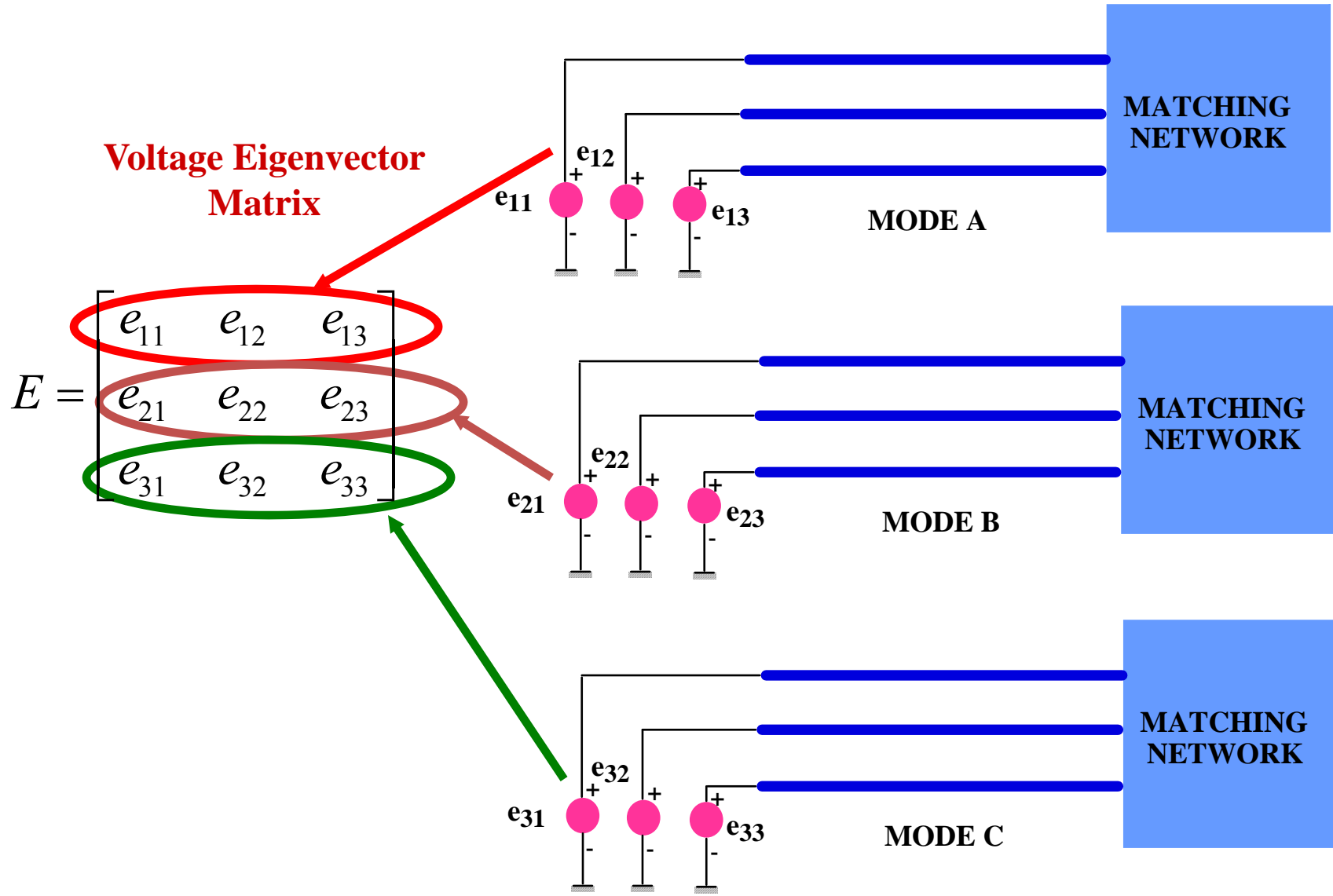
$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \quad \Lambda_m = \begin{bmatrix} \frac{1}{v_{m1}} & 0 & 0 \\ 0 & \frac{1}{v_{m2}} & 0 \\ 0 & 0 & \frac{1}{v_{m3}} \end{bmatrix}$$

$$HCLH^{-1} = \Lambda_m^2$$

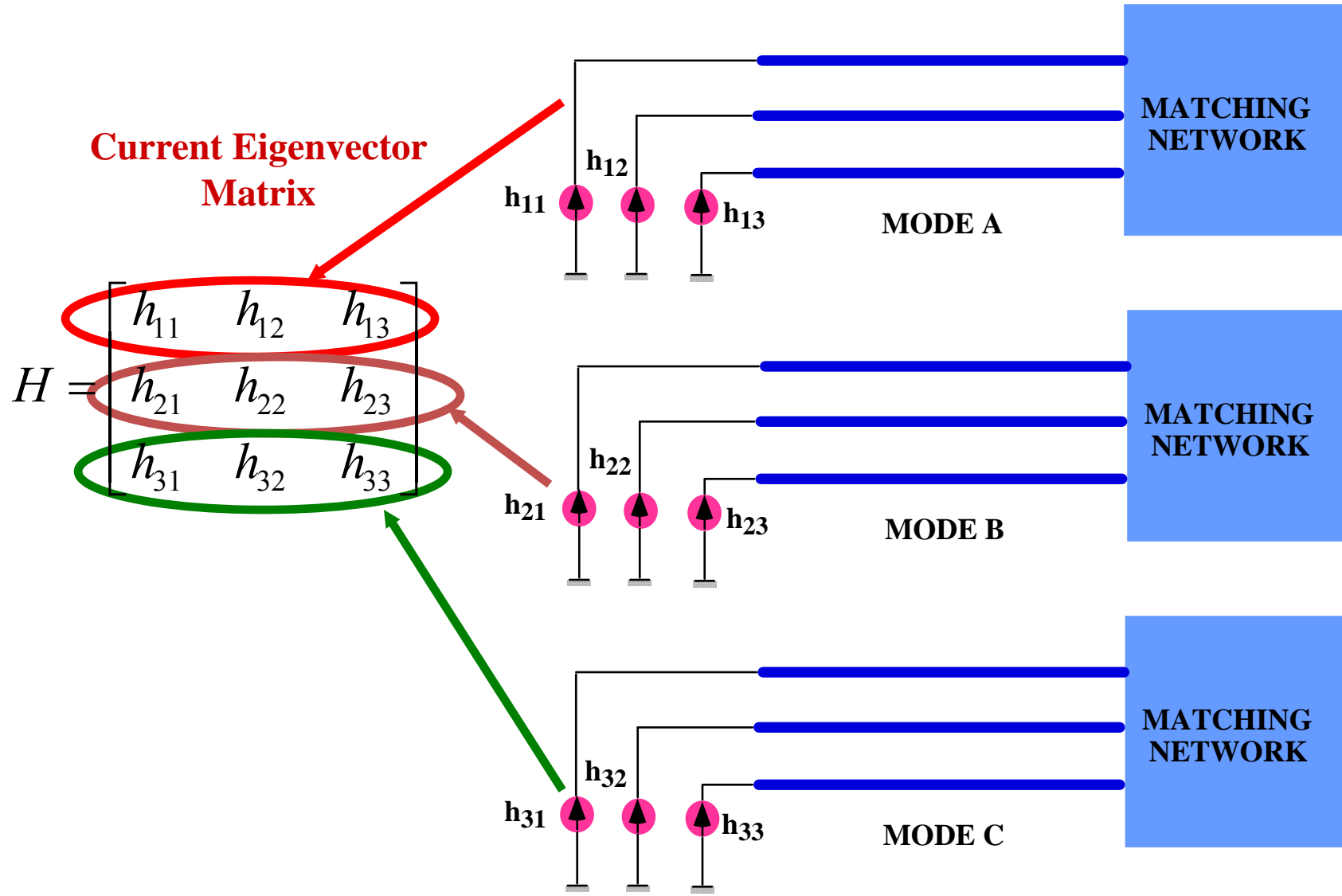
gives

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \quad \Lambda_m = \begin{bmatrix} \frac{1}{v_{m1}} & 0 & 0 \\ 0 & \frac{1}{v_{m2}} & 0 \\ 0 & 0 & \frac{1}{v_{m3}} \end{bmatrix}$$

Modal Voltage Excitation



Modal Current Excitation



EIGENVECTORS

E : voltage eigenvector matrix

H : current eigenvector matrix

E* and *H* depend on both *L* and *C

Special case: For a two-line *symmetric* system, *E* and *H* are equal and independent of *L* and *C*. The eigenvectors are [1,1] and [1,-1]

Line variables are recovered using

$$V = E^{-1}V_m$$

$$I = H^{-1}I_m$$

MODAL AND LINE IMPEDANCE MATRICES

By requiring $V_m = Z_m I_m$, we arrive at

$$Z_m = \Lambda_m^{-1} E L H^{-1}$$

Z_m is a diagonal impedance matrix which relates modal voltages to modal currents

By requiring $V = Z_c I$, one can define a line impedance matrix Z_c .
 Z_c relates line voltages to line currents

Z_m is diagonal. Z_c contains nonzero off-diagonal elements.
We can show that

$$Z_c = E^{-1} Z_m H = E^{-1} \Lambda_m^{-1} E L$$

$$\Lambda_m = \begin{bmatrix} \frac{1}{v_{m1}} & & & \\ & \frac{1}{v_{m2}} & & \\ & & \bullet & \\ & & & \frac{1}{v_{mn}} \end{bmatrix}$$

$$W(u) = \begin{bmatrix} e^{-\frac{j\omega u}{v_{m1}}} & & & \\ & e^{-\frac{j\omega u}{v_{m2}}} & & \\ & & \bullet & \\ & & & e^{-\frac{j\omega u}{v_{mn}}} \end{bmatrix}$$

MATRIX CONSTRUCTION

$$[L]_{ii} = L_s$$

$$[L]_{ij} = L_{ij}^{(m)}$$

$$[C]_{ii} = C_s + \sum_{j=1}^n C_{ij}^{(m)}$$

$$[C]_{ij} = -C_{ij}^{(m)}$$

$$[R]_{ii} = R_s$$

$$[R]_{ij} = R_{ij}^m$$

$$[G]_{ii} = G_s + \sum_{j=1}^n G_{ij}^{(m)}$$

$$[G]_{ij} = -G_{ij}^{(m)}$$

Modal Variables

$$V_m = EV \qquad I_m = HI$$

General solution in modal space is:

$$V_m(z) = W(z)A + W(-z)B$$

$$I_m(z) = Z_m^{-1} [W(z)A - W(-z)B]$$

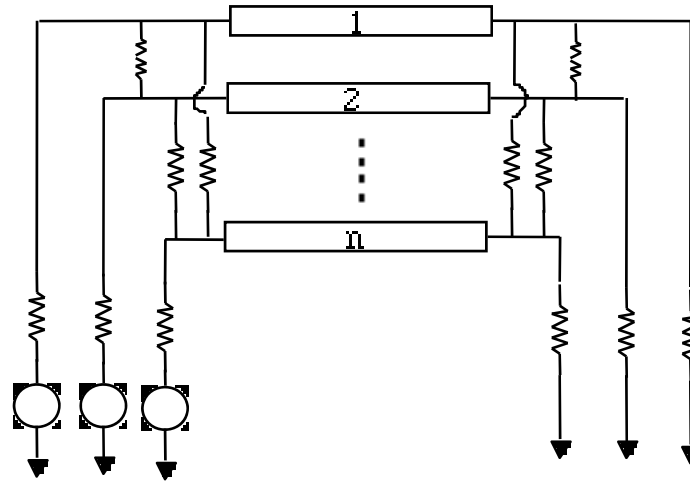
Modal impedance matrix

$$Z_m = \Lambda_m^{-1} E L H^{-1}$$

Line impedance matrix

$$Z_c = E^{-1} Z_m H = E^{-1} \Lambda_m^{-1} E L$$

N-Line Network



Z_s : Source impedance matrix

Z_L : Load impedance matrix

V_s : Source vector

Reflection Coefficients

Source reflection coefficient matrix

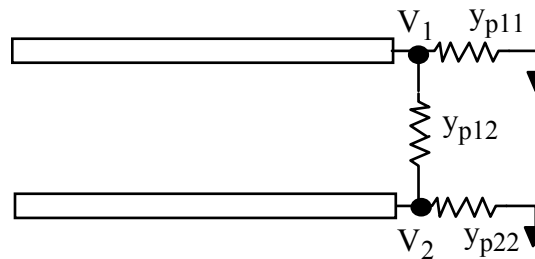
$$\Gamma_s = [1_n + EZ_s L^{-1} E^{-1} \Lambda_m]^{-1} [1_n - EZ_s L^{-1} E^{-1} \Lambda_m]$$

Load reflection coefficient matrix

$$\Gamma_L = [1_n + EZ_L L^{-1} E^{-1} \Lambda_m]^{-1} [1_n - EZ_L L^{-1} E^{-1} \Lambda_m]$$

- * **The reflection and transmission coefficient are $n \times n$ matrices with off-diagonal elements. The off-diagonal elements account for the coupling between modes.**
- * **Z_s and Z_L can be chosen so that the reflection coefficient matrices are zero**

TERMINATION NETWORK



$$I_1 = y_{p11} V_1 + y_{p12}(V_1 - V_2)$$

$$I_2 = y_{p22} V_2 + y_{p12}(V_2 - V_1)$$

In general for a multiline system

$$\mathbf{I} = \mathbf{YV} \Rightarrow \mathbf{V} = \mathbf{ZI} \quad \mathbf{Z} = \mathbf{Y}^{-1}$$

$$\text{Also, } z_{ij} \neq \frac{1}{y_{ij}}$$

Note : $y_{ii} \neq y_{pii}$, $y_{ij} = -y_{ji}$, ($i \neq j$)

Termination Network Construction

To get impedance matrix Z

- Get physical impedance values y_{pij}
- Calculate y_{ij} 's from y_{pij} 's
- Construct Y matrix
- Invert Y matrix to obtain Z matrix

Remark: If $y_{pij} = 0$ for all $i \neq j$, then $Y = Z^{-1}$ and $z_{ii} = 1/y_{ii}$

Procedure for Multiconductor Solution

- 1) Get L and C matrices and calculate LC product
- 2) Get square root of eigenvalues and eigenvectors of LC matrix $\rightarrow \Lambda_m$
- 3) Arrange eigenvectors into the voltage eigenvector matrix E
- 4) Get square root of eigenvalues and eigenvectors of CL matrix $\rightarrow \Lambda_m$
- 5) Arrange eigenvectors into the current eigenvector matrix H
- 6) Invert matrices E, H, Λ_m .
- 7) Calculate the line impedance matrix $Z_c \rightarrow Z_c = E^{-1}\Lambda_m^{-1}EL$

Procedure for Multiconductor Solution

- 8) Construct source and load impedance matrices $Z_s(t)$ and $Z_L(t)$
- 9) Construct source and load reflection coefficient matrices $\Gamma_1(t)$ and $\Gamma_2(t)$.
- 10) Construct source and load transmission coefficient matrices $T_1(t)$, $T_2(t)$
- 11) Calculate modal voltage sources $g_1(t)$ and $g_2(t)$
- 12) Calculate modal voltage waves:

$$\mathbf{a}_1(t) = \mathbf{T}_1(t)\mathbf{g}_1(t) + \Gamma_1(t)\mathbf{a}_2(t - \tau_m)$$

$$\mathbf{a}_2(t) = \mathbf{T}_2(t)\mathbf{g}_2(t) + \Gamma_2(t)\mathbf{a}_1(t - \tau_m)$$

$$\mathbf{b}_1(t) = \mathbf{a}_2(t - \tau_m)$$

$$\mathbf{b}_2(t) = \mathbf{a}_1(t - \tau_m)$$

Wave-Shifting Solution

$$\mathbf{a}_i(t - \tau_m) = \begin{bmatrix} a_{i-\text{mode}-1}(t - \tau_{m1}) \\ a_{i-\text{mode}-2}(t - \tau_{m2}) \\ \bullet \\ a_{i-\text{mode}-n}(t - \tau_{mn}) \end{bmatrix}$$

τ_{mi} is the delay associated with mode i . $\tau_{mi} = \text{length}/\text{velocity}$ of mode i . The modal voltage wave vectors $\mathbf{a}_1(t)$ and $\mathbf{a}_2(t)$ need to be stored since they contain the history of the system.

13) Calculate total modal voltage vectors:

$$\mathbf{V}_{m1}(t) = \mathbf{a}_1(t) + \mathbf{b}_1(t)$$

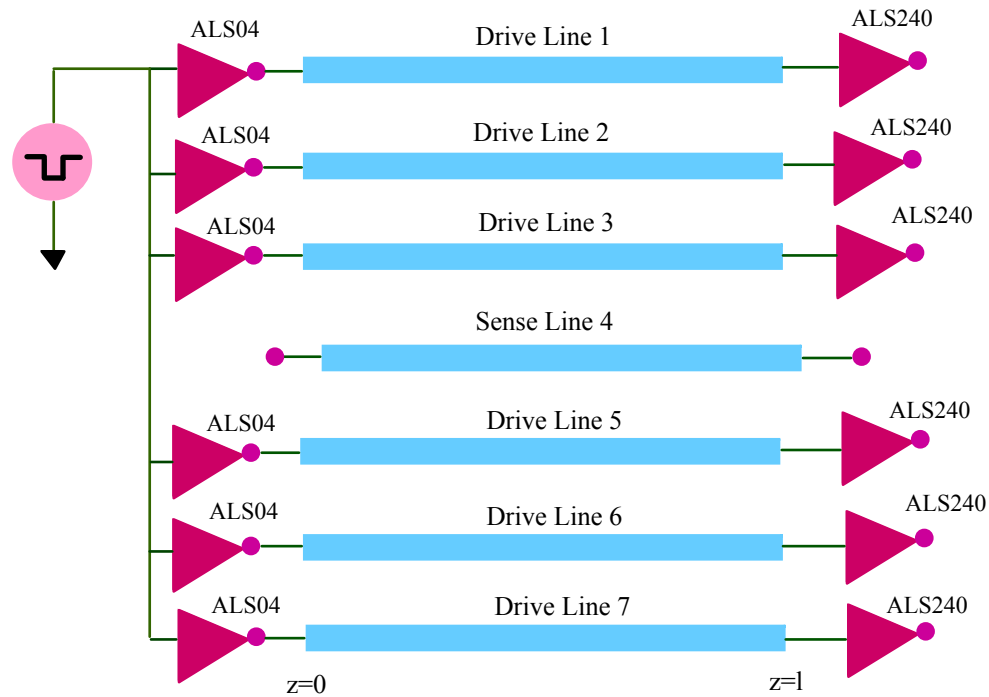
$$\mathbf{V}_{m2}(t) = \mathbf{a}_2(t) + \mathbf{b}_2(t)$$

14) Calculate line voltage vectors:

$$\mathbf{V}_1(t) = \mathbf{E}^{-1}\mathbf{V}_{m1}(t)$$

$$\mathbf{V}_2(t) = \mathbf{E}^{-1}\mathbf{V}_{m2}(t)$$

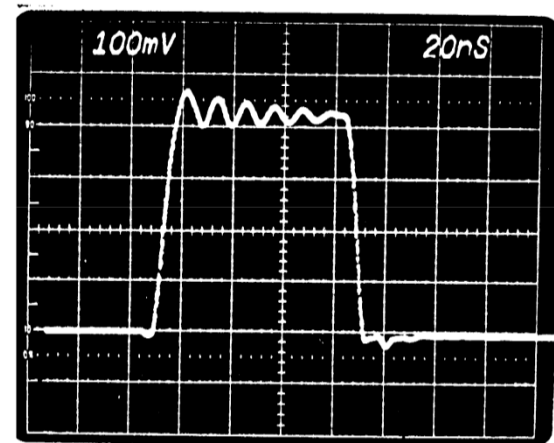
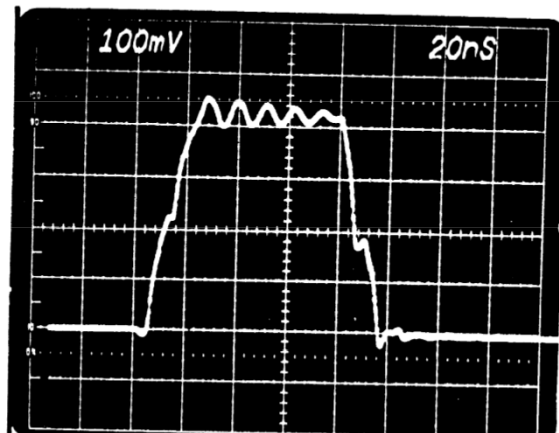
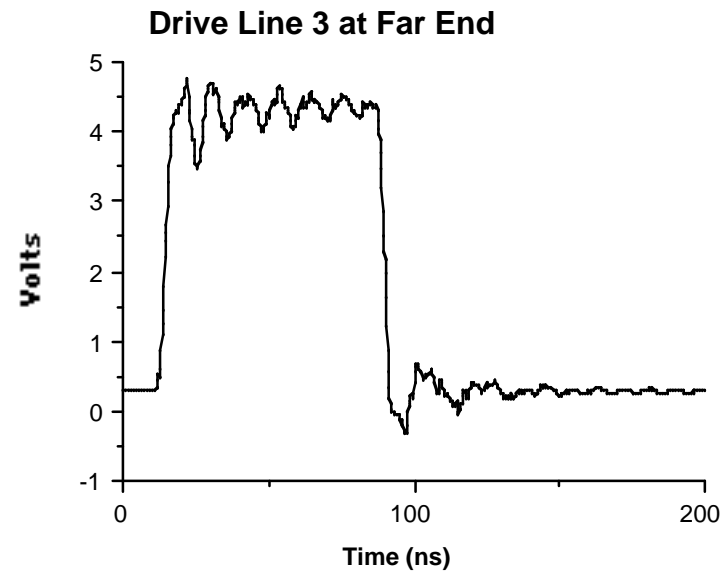
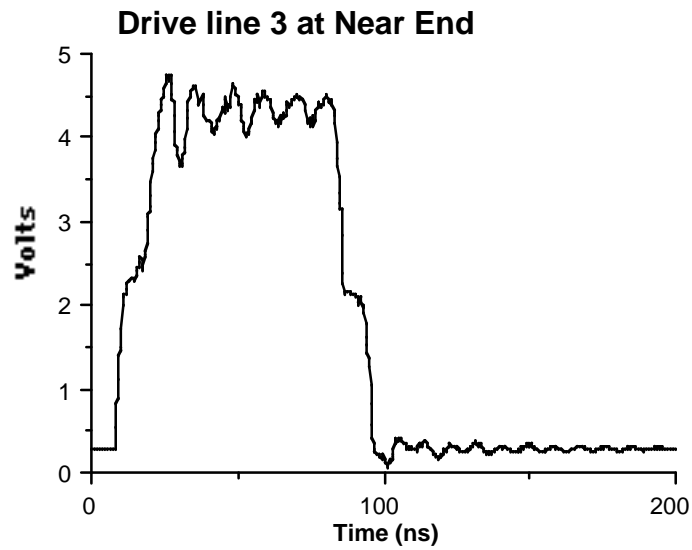
7-Line Coupled-Microstrip System



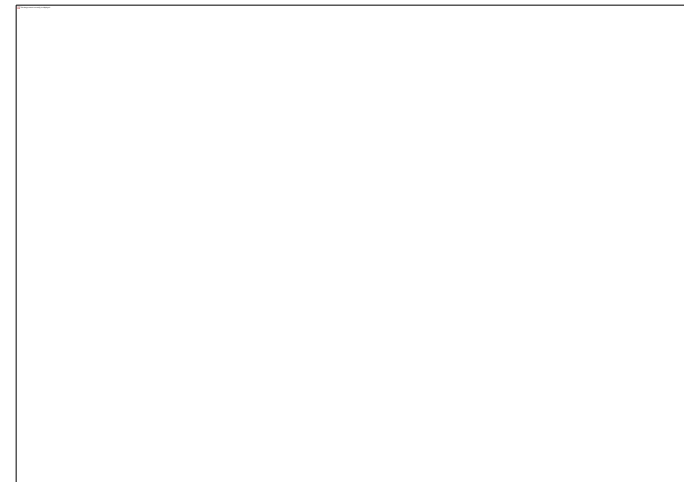
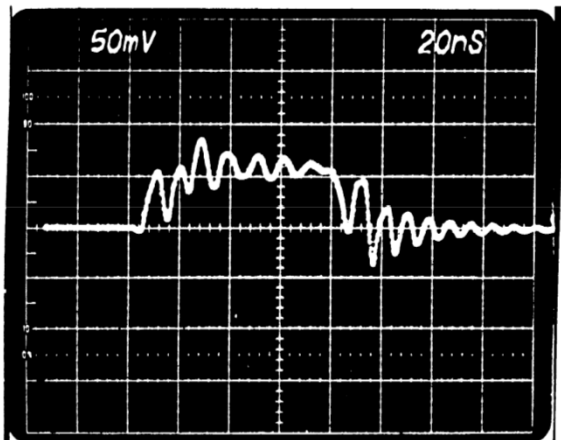
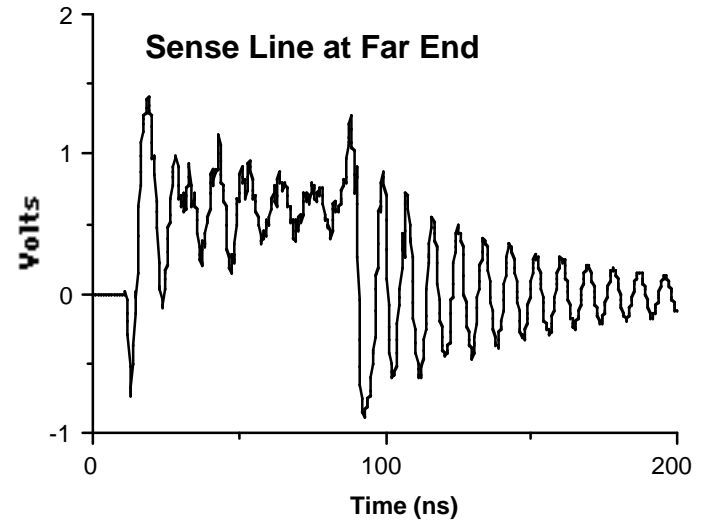
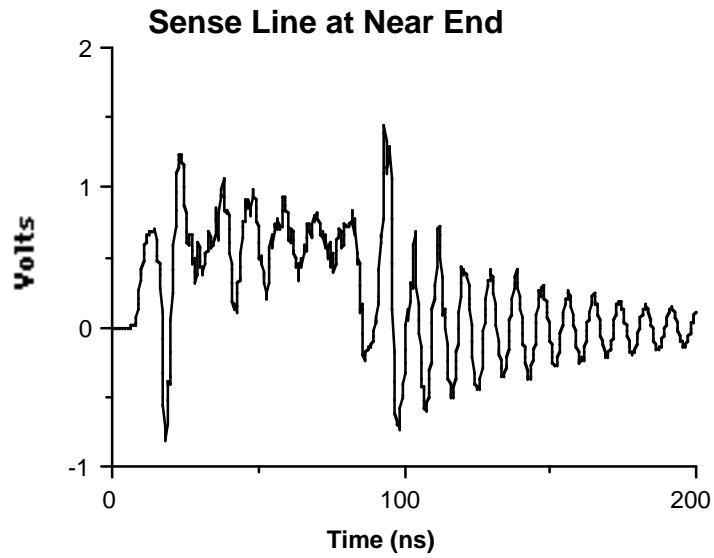
$$L_s = 312 \text{ nH/m}; \quad C_s = 100 \text{ pF/m};$$

$$L_m = 85 \text{ nH/m}; \quad C_m = 12 \text{ pF/m}.$$

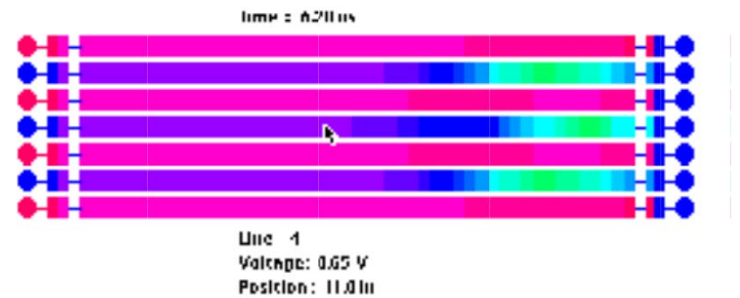
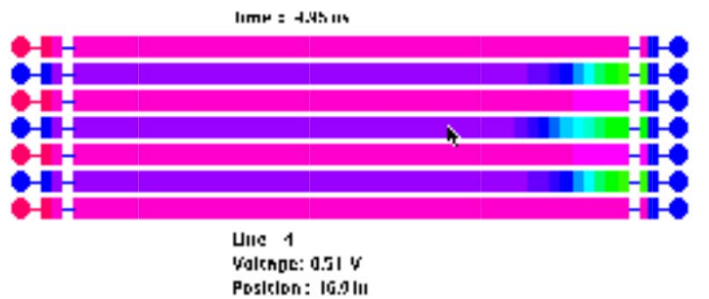
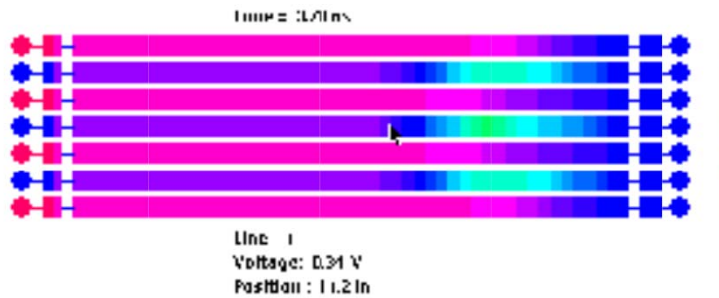
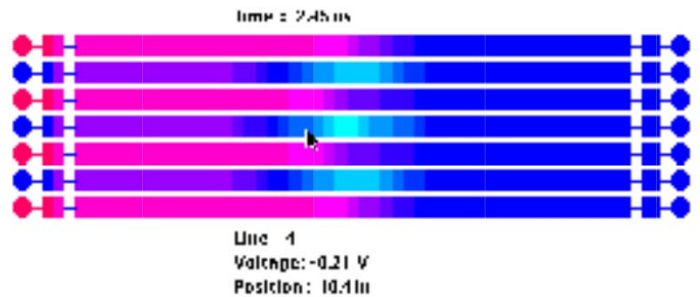
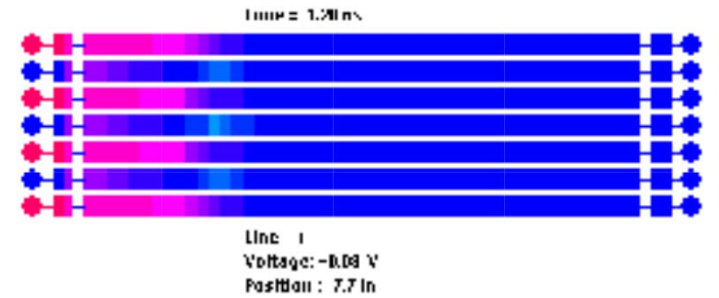
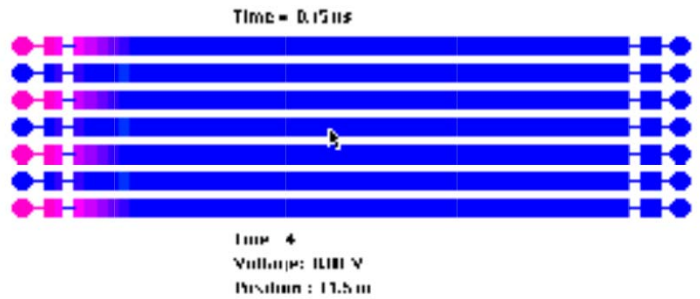
Drive Line 3



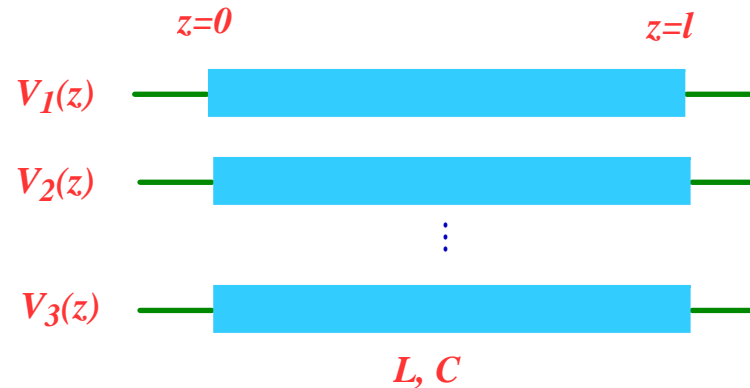
Sense Line



Multiconductor Simulation



LOSSY COUPLED TRANSMISSION LINES

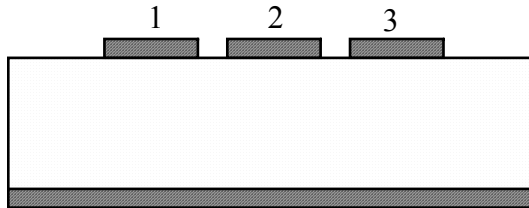


$$-\frac{\partial V}{\partial z} = RI + L \frac{\partial I}{\partial t}$$

$$-\frac{\partial I}{\partial z} = GV + C \frac{\partial V}{\partial t}$$

*Solution is best found using a numerical approach
(See References)*

Three-Line Microstrip



$$-\frac{\partial V_1}{\partial z} = L_{11} \frac{\partial I_1}{\partial t} + L_{12} \frac{\partial I_2}{\partial t} + L_{13} \frac{\partial I_3}{\partial t}$$

$$-\frac{\partial I_1}{\partial z} = C_{11} \frac{\partial V_1}{\partial t} + C_{12} \frac{\partial V_2}{\partial t} + C_{13} \frac{\partial V_3}{\partial t}$$

$$-\frac{\partial V_2}{\partial z} = L_{21} \frac{\partial I_1}{\partial t} + L_{22} \frac{\partial I_2}{\partial t} + L_{23} \frac{\partial I_3}{\partial t}$$

$$-\frac{\partial I_2}{\partial z} = C_{21} \frac{\partial V_1}{\partial t} + C_{22} \frac{\partial V_2}{\partial t} + C_{23} \frac{\partial V_3}{\partial t}$$

$$-\frac{\partial V_3}{\partial z} = L_{31} \frac{\partial I_1}{\partial t} + L_{32} \frac{\partial I_2}{\partial t} + L_{33} \frac{\partial I_3}{\partial t}$$

$$-\frac{\partial I_3}{\partial z} = C_{31} \frac{\partial V_1}{\partial t} + C_{32} \frac{\partial V_2}{\partial t} + C_{33} \frac{\partial V_3}{\partial t}$$

Three-Line – Alpha Mode

Subtract (1c) from (1a) and (2c) from (2a), we get

$$\begin{aligned} -\frac{\partial V_\alpha}{\partial z} &= (L_{11} - L_{13}) \frac{\partial I_\alpha}{\partial t} \\ -\frac{\partial I_\alpha}{\partial z} &= (C_{11} - C_{13}) \frac{\partial V_\alpha}{\partial t} \end{aligned}$$

This defines the Alpha mode with:

$$V_\alpha = V_1 - V_3 \quad \text{and} \quad I_\alpha = I_1 - I_3$$

The wave impedance of that mode is:

$$Z_\alpha = \sqrt{\frac{L_{11} - L_{13}}{C_{11} - C_{13}}}$$

and the velocity is

$$u_\alpha = \frac{1}{\sqrt{(L_{11} - L_{13})(C_{11} - C_{13})}}$$

Three-Line – Modal Decomposition

In order to determine the next mode, assume that

$$V_\beta = V_1 + \beta V_2 + V_3$$

$$I_\beta = I_1 + \beta I_2 + I_3$$

$$-\frac{\partial V_\beta}{\partial z} = (L_{11} + \beta L_{21} + L_{31}) \frac{\partial I_1}{\partial t} + (L_{12} + \beta L_{22} + L_{32}) \frac{\partial I_2}{\partial t} + (L_{13} + \beta L_{23} + L_{33}) \frac{\partial I_3}{\partial t}$$
$$-\frac{\partial I_\beta}{\partial z} = (C_{11} + \beta C_{21} + C_{31}) \frac{\partial V_1}{\partial t} + (C_{12} + \beta C_{22} + C_{32}) \frac{\partial V_2}{\partial t} + (C_{13} + \beta C_{23} + C_{33}) \frac{\partial V_3}{\partial t}$$

By reciprocity $L_{32} = L_{23}$, $L_{21} = L_{12}$, $L_{13} = L_{31}$

By symmetry, $L_{12} = L_{23}$

Also by approximation, $L_{22} \approx L_{11}$, $L_{11} + L_{13} \approx L_{11}$

Three-Line – Modal Decomposition

$$-\frac{\partial V_{\beta}}{\partial z} = (L_{11} + \beta L_{12} + L_{13}) \left(\frac{\partial I_1}{\partial t} + \frac{\partial I_3}{\partial t} \right) + (2L_{12} + \beta L_{11}) \frac{\partial I_2}{\partial t}$$

In order to balance the right-hand side into I_{β} , we need to have

$$(2L_{12} + \beta L_{11}) I_2 = \beta (L_{11} + \beta L_{12} + L_{13}) I_2 \approx \beta (L_{11} + \beta L_{12}) I_2$$

$$2L_{12} = \beta^2 L_{12}$$

or $\beta = \pm\sqrt{2}$

Therefore the other two modes are defined as

The Beta mode with

Three-Line – Beta Mode

The Beta mode with

$$V_{\beta} = V_1 + \sqrt{2}V_2 + V_3$$

$$I_{\beta} = I_1 + \sqrt{2}I_2 + I_3$$

The characteristic impedance of the Beta mode is:

$$Z_{\beta} = \sqrt{\frac{L_{11} + \sqrt{2}L_{12} + L_{13}}{C_{11} + \sqrt{2}C_{12} + C_{13}}}$$

and propagation velocity of the Beta mode is

$$u_{\beta} = \frac{1}{\sqrt{(L_{11} + \sqrt{2}L_{12} + L_{13})(C_{11} + \sqrt{2}C_{12} + C_{13})}}$$

Three-Line – Delta Mode

The Delta mode is defined such that

$$V_{\delta} = V_1 - \sqrt{2}V_2 + V_3$$

$$I_{\delta} = I_1 - \sqrt{2}I_2 + I_3$$

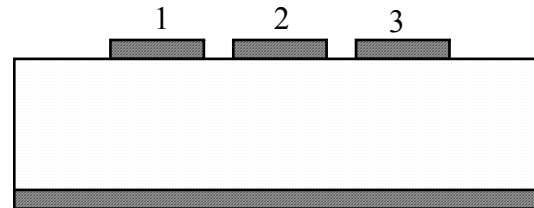
The characteristic impedance of the Delta mode is

$$Z_{\delta} = \sqrt{\frac{L_{11} - \sqrt{2}L_{12} + L_{13}}{C_{11} - \sqrt{2}C_{12} + C_{13}}}$$

The propagation velocity of the Delta mode is:

$$u_{\delta} = \frac{1}{\sqrt{(L_{11} - \sqrt{2}L_{12} + L_{13})(C_{11} - \sqrt{2}C_{12} + C_{13})}}$$

Symmetric 3-Line Microstrip



In summary: we have 3 modes for the 3-line system

$$E = \begin{pmatrix} 1 & 0 & -1 \\ 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

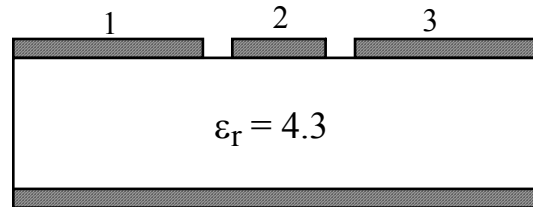
Alpha mode

Beta mode*

Delta mode*

*neglecting coupling between nonadjacent lines

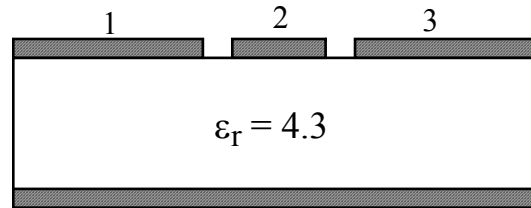
Coplanar Waveguide



$$L(nH / m) = \begin{pmatrix} 346 & 162 & 67 \\ 152 & 683 & 152 \\ 67 & 162 & 346 \end{pmatrix} \quad C(pF / m) = \begin{pmatrix} 113 & 17 & 5 \\ 16 & 53 & 16 \\ 5 & 17 & 113 \end{pmatrix}$$

$$E = \begin{pmatrix} 0.45 & 0.12 & 0.45 \\ 0.5 & 0 & -0.5 \\ -0.45 & 0.87 & -0.45 \end{pmatrix} \quad H = \begin{pmatrix} 0.44 & 0.49 & 0.44 \\ 0.5 & 0 & -0.5 \\ -0.10 & 0.88 & -0.10 \end{pmatrix}$$

Coplanar Waveguide

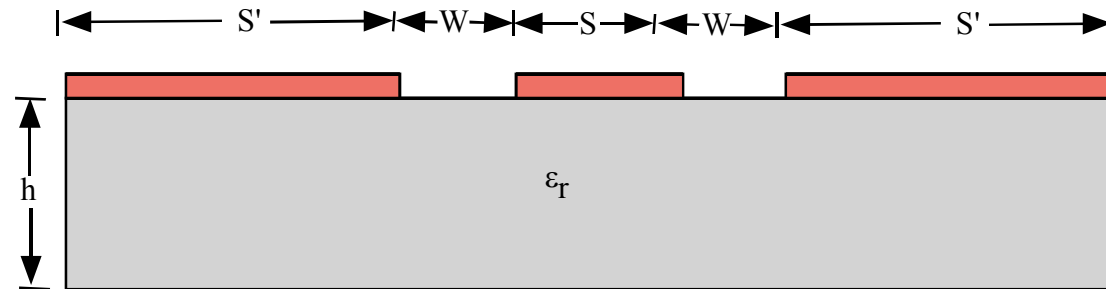


$$Z_m(\Omega) = \begin{pmatrix} 73 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & 94 \end{pmatrix}$$

$$Z_c(\Omega) = \begin{pmatrix} 56 & 23 & 8 \\ 22 & 119 & 22 \\ 8 & 23 & 56 \end{pmatrix}$$

$$v_p(m/ns) = \begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0.17 & 0 \\ 0 & 0 & 0.18 \end{pmatrix}$$

Coplanar Waveguide



$K(k)$: Complete Elliptic Integral of the first kind

$$k = \frac{S}{S + 2W}$$

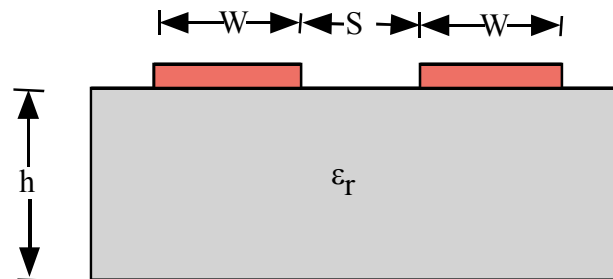
$$Z_{ocp} = \frac{30\pi}{\sqrt{\frac{\epsilon_r + 1}{2}}} \frac{K'(k)}{K(k)} \text{ (ohm)}$$

$$K'(k) = K(k')$$

$$k' = (1 - k^2)^{1/2}$$

$$v_{cp} = \left(\frac{2}{\epsilon_r + 1} \right)^{1/2} c$$

Coplanar Strips



$$Z_{ocs} = \frac{120\pi}{\sqrt{\frac{\epsilon_r + 1}{2}}} \frac{K'(k)}{K(k)} \text{ (ohm)}$$

Qualitative Comparison

Characteristic	Microstrip	Coplanar Wguide	Coplanar strips
ϵ_{eff}^*	~6.5	~5	~5
Power handling	High	Medium	Medium
Radiation loss	Low	Medium	Medium
Unloaded Q	High	Medium	Low or High
Dispersion	Small	Medium	Medium
Mounting (shunt)	Hard	Easy	Easy
Mounting (series)	Easy	Easy	Easy

* Assuming $\epsilon_r=10$ and $h=0.025$ inch