

ECE 546

Lecture -16

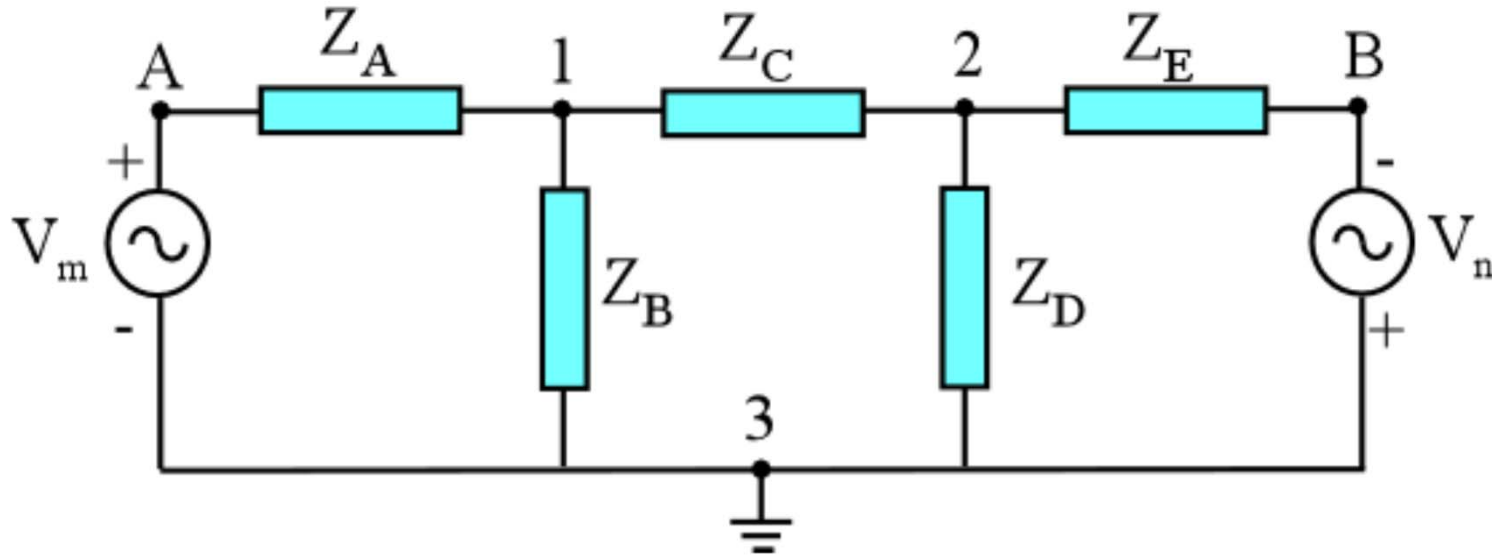
MNA and SPICE

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Nodal Analysis

The Node Voltage method consists in determining potential differences between nodes and ground (reference) using KCL



For Node 1:
$$\frac{V_1 - V_m}{Z_A} + \frac{V_1}{Z_B} + \frac{V_1 - V_2}{Z_C} = 0$$

Nodal Analysis

For Node 2:
$$\frac{V_2 - V_1}{Z_C} + \frac{V_2}{Z_D} + \frac{V_2 - V_n}{Z_E} = 0$$

Rearranging the terms gives:

$$\left(\frac{1}{Z_A} + \frac{1}{Z_B} + \frac{1}{Z_C} \right) V_1 - \left(\frac{1}{Z_C} \right) V_2 = \left(\frac{1}{Z_A} \right) V_m$$

$$-\left(\frac{1}{Z_C} \right) V_1 + \left(\frac{1}{Z_C} + \frac{1}{Z_D} + \frac{1}{Z_E} \right) V_2 = -\left(\frac{1}{Z_E} \right) V_n$$

Defining:
$$G_A = \frac{1}{Z_A}, \quad G_B = \frac{1}{Z_B}, \quad G_C = \frac{1}{Z_C}, \quad G_D = \frac{1}{Z_D}, \quad G_E = \frac{1}{Z_E}$$

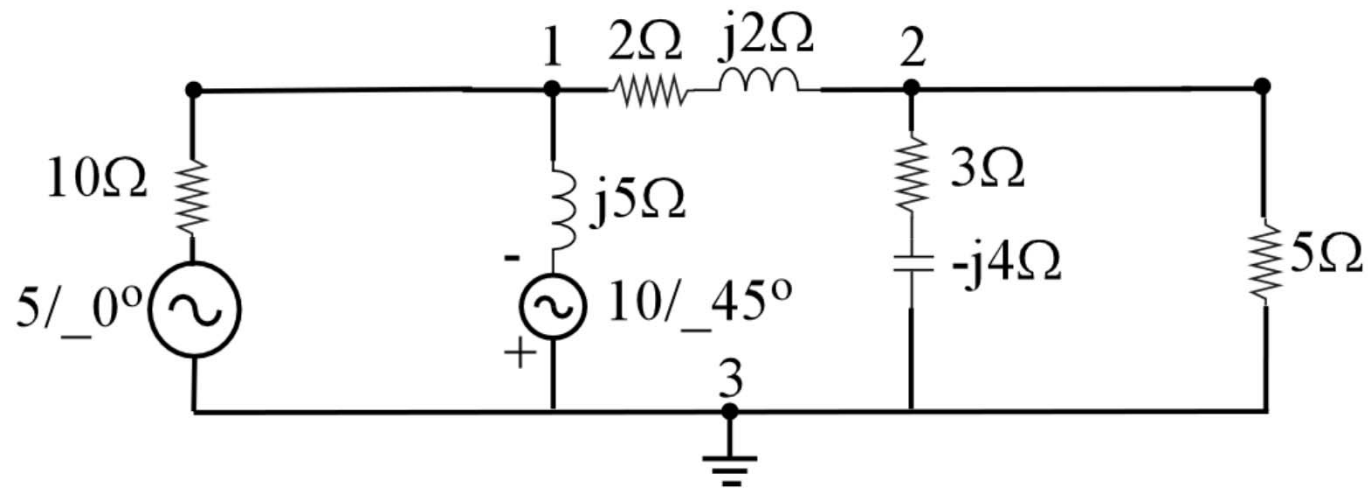
Nodal Analysis

Rearranging the terms gives:

$$\begin{bmatrix} (G_A + G_B + G_C) & -G_C \\ -G_C & (G_C + G_D + G_E) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} G_A V_m \\ -G_E V_m \end{bmatrix}$$

The system can be solved to yield V_1 and V_2 .

Nodal Analysis



For Node 1:

$$\frac{V_1 - 5\angle 0^\circ}{10} + \frac{V_1 + j10\angle 45^\circ}{j5} + \frac{V_1 - V_2}{2 + j2} = 0$$

For Node 2:

$$\frac{V_2 - V_1}{2 + j2} + \frac{V_2}{3 - j4} + \frac{V_2}{5} = 0$$

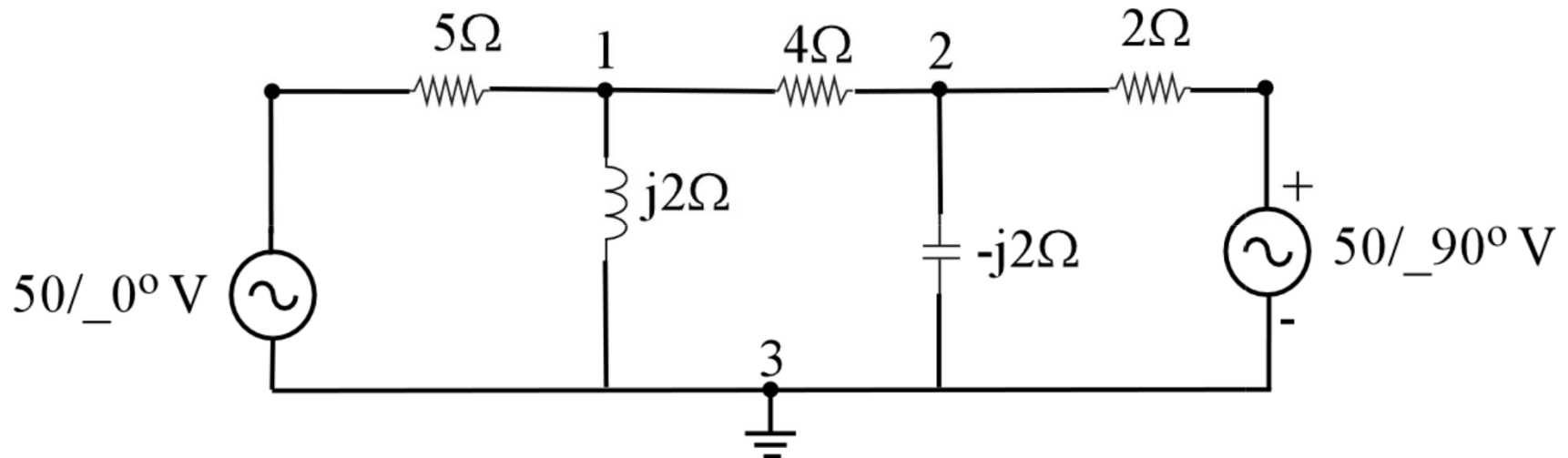
Nodal Analysis

Rearranging the terms gives:

$$\left(\frac{1}{10} + \frac{1}{j5} + \frac{1}{2 + j2}\right)V_1 - \left(\frac{1}{2 + j2}\right)V_2 = \frac{5\angle 0^\circ}{10} - \frac{10\angle 45^\circ}{j5}$$

$$-\left(\frac{1}{2 + j2}\right)V_1 + \left(\frac{1}{2 + j2} + \frac{1}{3 - j4} + \frac{1}{5}\right)V_2 = 0$$

Nodal Analysis



$$\begin{bmatrix} \left(\frac{1}{5} + \frac{1}{j2} + \frac{1}{4}\right) & -\left(\frac{1}{4}\right) \\ -\left(\frac{1}{4}\right) & \left(\frac{1}{4} + \frac{1}{-j2} + \frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{50\angle 0^\circ}{5} \\ \frac{50\angle 90^\circ}{5} \end{bmatrix}$$

[Y]

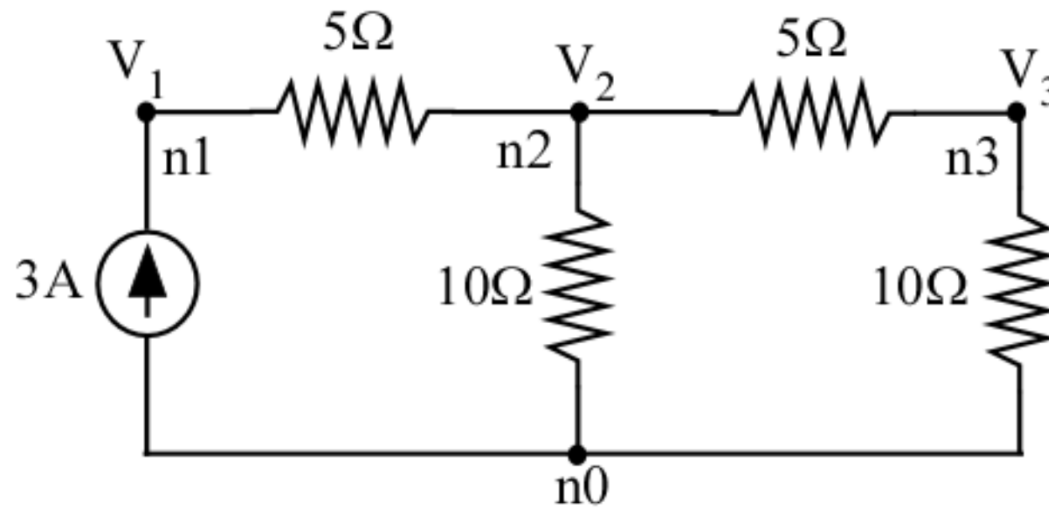
[v]=[i]

Nodal Analysis - Solution

$$V_1 = \frac{\begin{vmatrix} 10 & 0.25 \\ j25 & 0.75 + j0.5 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ 0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{13.5 \angle 56.3^\circ}{0.546 \angle -15.95^\circ} = 24.7 \angle 72.25^\circ \text{ V}$$

$$V_1 = \frac{\begin{vmatrix} 0.45 - j0.5 & 10 \\ -0.25 & j25 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ 0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{18.35 \angle 37.8^\circ}{0.546 \angle -15.95^\circ} = 33.6 \angle 53.75^\circ \text{ V}$$

Nodal Formulation



$$\text{Node 1: } -3 + \frac{V_1 - V_2}{5} = 0$$

$$\text{Node 2: } \frac{V_2 - V_1}{5} + \frac{V_2}{10} + \frac{V_2 - V_3}{5} = 0$$

$$\text{Node 3: } \frac{V_3 - V_2}{5} + \frac{V_3}{10} = 0$$

Nodal Solution

Arrange in matrix form: $[G][V]=[I]$

$$\begin{bmatrix} 0.2 & -0.2 & 0 \\ -0.2 & 0.5 & -0.2 \\ 0 & -0.2 & 0.3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Use Gaussian elimination to form an upper triangular matrix

$$\begin{bmatrix} 0.2 & -0.2 & 0 \\ 0 & 0.3 & -0.2 \\ 0 & 0 & 0.25 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

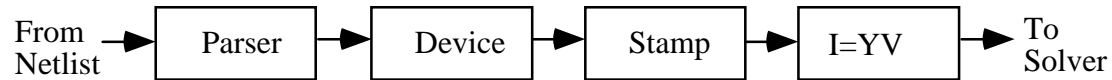
Solve for V_1 , V_2 and V_3 using backward substitution

This can always be solved no matter how large the matrix is

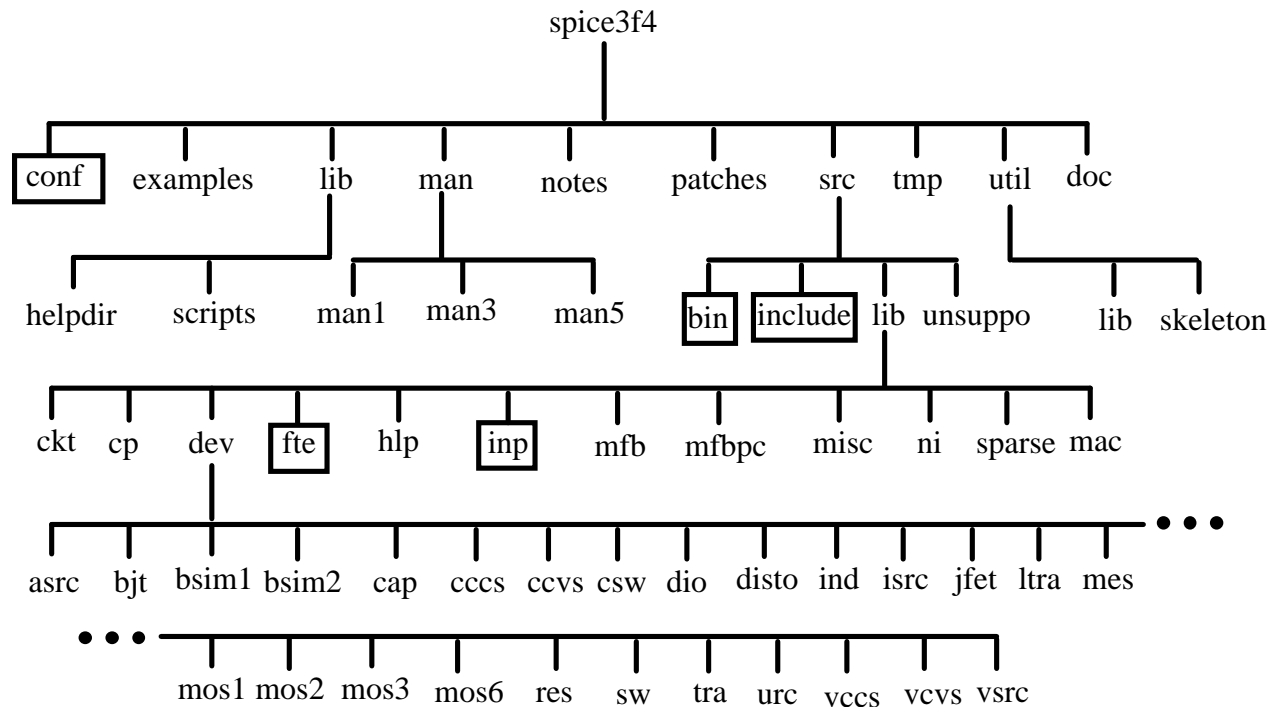
Why SPICE ?

- ❑ Established platform
- ❑ Powerful engine
- ❑ Source code available for free
- ❑ Extensive libraries of devices
- ❑ New device installation procedure easy

SPICE



SPICE Directory Structure



MOS SPICE Parameters

Symbol	Description	Value	Units
L_{drawn}	Device length (drawn)	0.35	μm
L_{eff}	Device length (effective)	0.25	μm
t_{ox}	Gate oxide thickness	70	Å
N_a	Density of acceptor ions in NFET channel	1.0×10^{17}	cm^{-3}
N_d	Density of donor ions in PFET channel	2.5×10^{17}	cm^{-3}
V_{Tn}	NFET threshold voltage	0.5	V
V_{Tp}	PFET threshold voltage	-0.5	V
λ	Channel modulation parameter	0.1	V^{-1}
γ	Body effect parameter	0.3	$\text{V}^{1/2}$
V_{sat}	Saturation velocity	1.7×10^5	m/s
μ_n	Electron mobility	400	cm^2/Vs
μ_p	Hole mobility	100	cm^2/Vs
k_n	NFET process transconductance	200	$\mu\text{A}/\text{V}^2$
k_p	PFET process transconductance	50	$\mu\text{A}/\text{V}^2$
C_{ox}	Gate oxide capacitance per unit area	5	$\text{fF}/\mu\text{m}^2$
C_{GSO}, C_{GDO}	Gate source and drain overlap capacitance	0.1	$\text{fF}/\mu\text{m}$
C_J	Junction capacitance	0.5	$\text{fF}/\mu\text{m}^2$
C_{JSW}	Junction sidewall capacitance	0.2	$\text{fF}/\mu\text{m}$
R_{poly}	Gate sheet resistance	4	Ω/square
R_{diff}	Source and drain sheet resistance	4	Ω/square

Problems

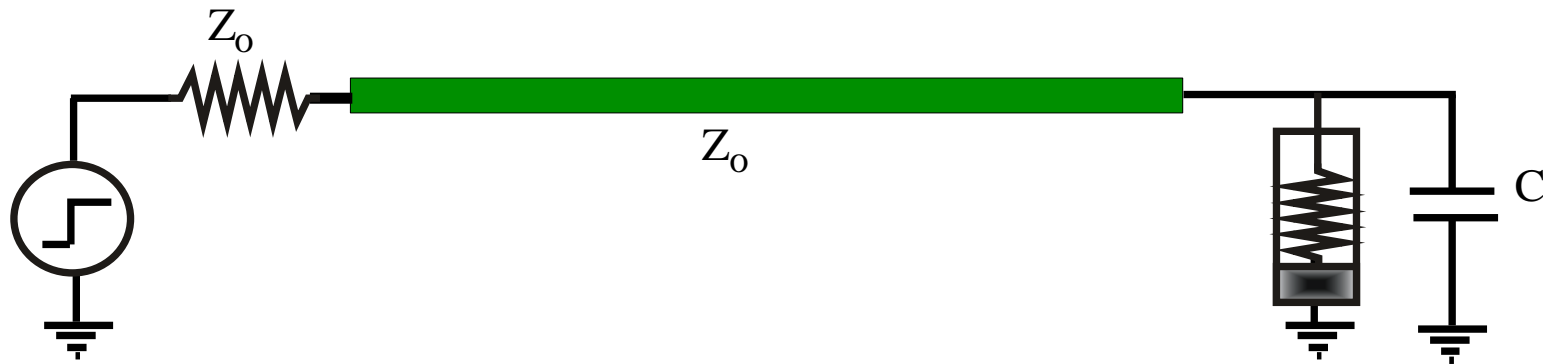
- **Nonlinear Devices**

- Diodes, transistors cannot be simulated in the frequency domain
- Capacitors and inductors are best described in the frequency domain
- Use time-domain representation for reactive elements (capacitors and inductors)

- **Circuit Size**

- Matrix size becomes prohibitively large

Motivations



- Loads are nonlinear
- Need to model reactive elements in the time domain
- Generalize to nonlinear reactive elements

Time-Domain Model for Linear Capacitor

For linear capacitor C with voltage v and current i which must satisfy

$$i = C \frac{dv}{dt}$$

Using the backward Euler scheme, we discretize time and voltage variables and obtain at time $t = nh$

$$v_{n+1} = v_n + hv'_{n+1}$$

Time-Domain Model for Linear Capacitor

After substitution, we obtain

$$v'_{n+1} = \frac{i_{n+1}}{C}$$

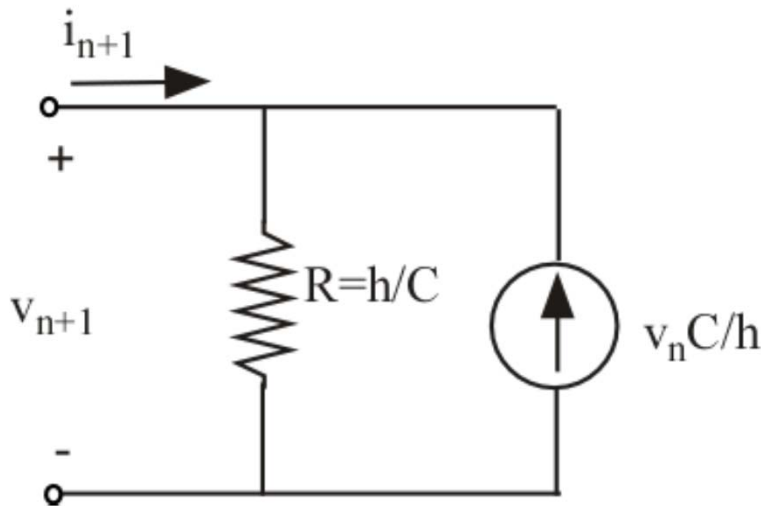
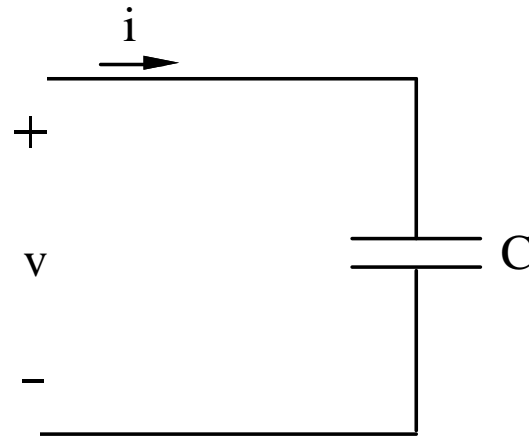
so that

$$v_{n+1} = v_n + h \frac{i_{n+1}}{C}$$

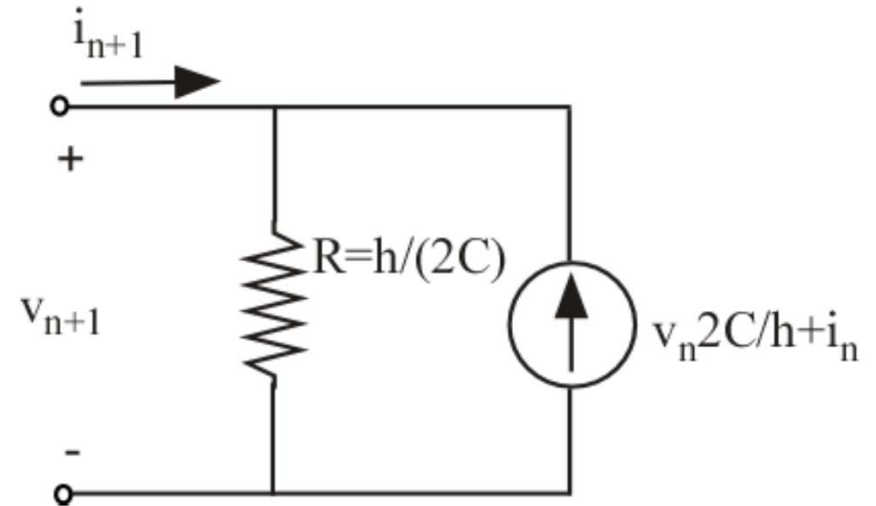
The solution for the current at t_{n+1} is, therefore,

$$i_{n+1} = \frac{C}{h} v_{n+1} - \frac{C}{h} v_n$$

Time-Domain Model for Linear Capacitor



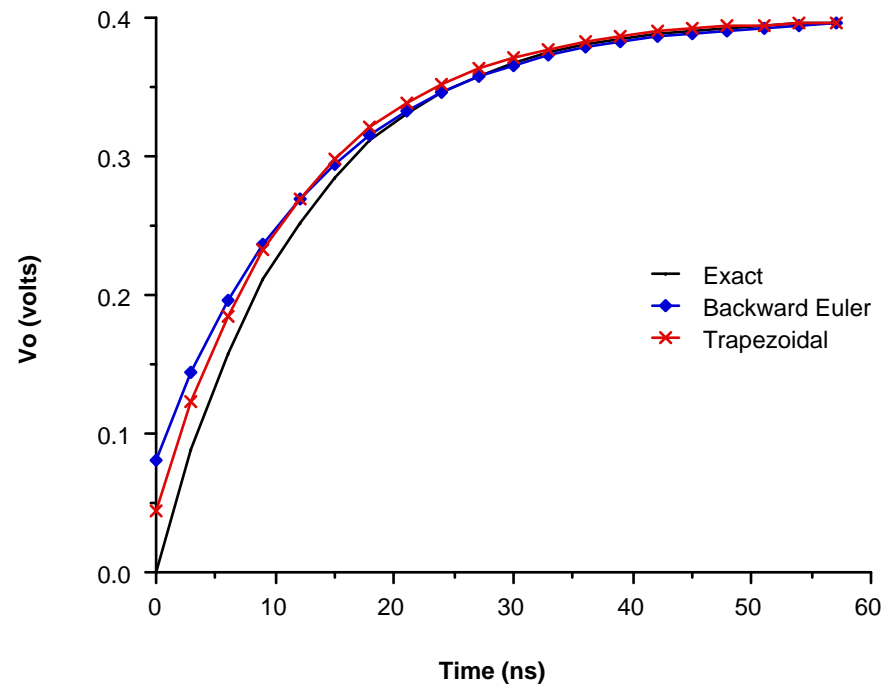
Backward Euler companion model at $t=nh$



Trapezoidal companion model at $t=nh$

Time-Domain Model for Linear Capacitor

Step response comparisons



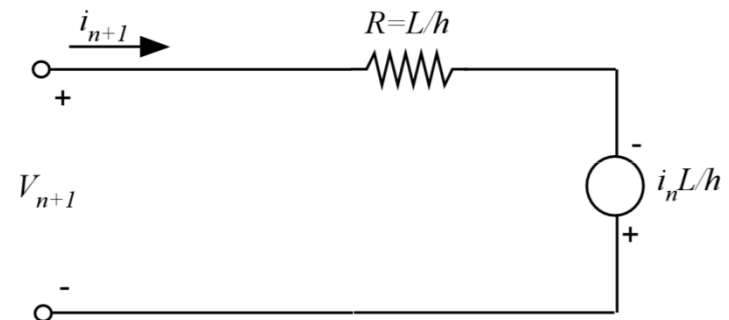
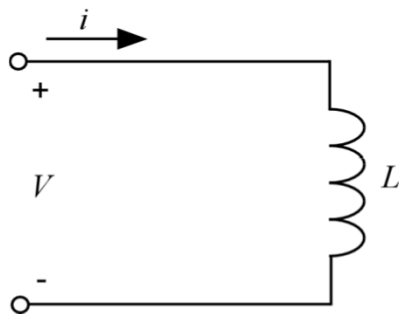
Time-Domain Model for Linear Inductor

$$v = L \frac{di}{dt}$$

$$\text{Backward Euler: } i_{n+1} = i_n + h i'_{n+1}$$

$$i'_{n+1} = \frac{v_{n+1}}{L}$$

$$v_{n+1} = \frac{L}{h} i_{n+1} - \frac{L}{h} i_n$$

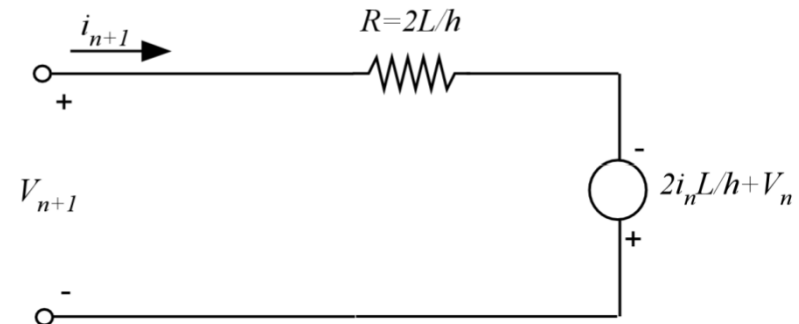
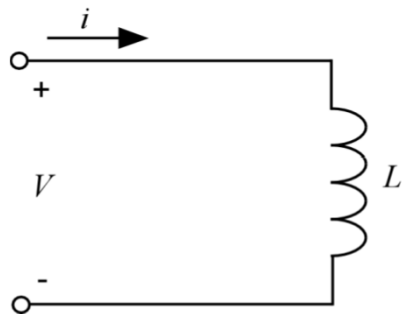


Time-Domain Model for Linear Inductor

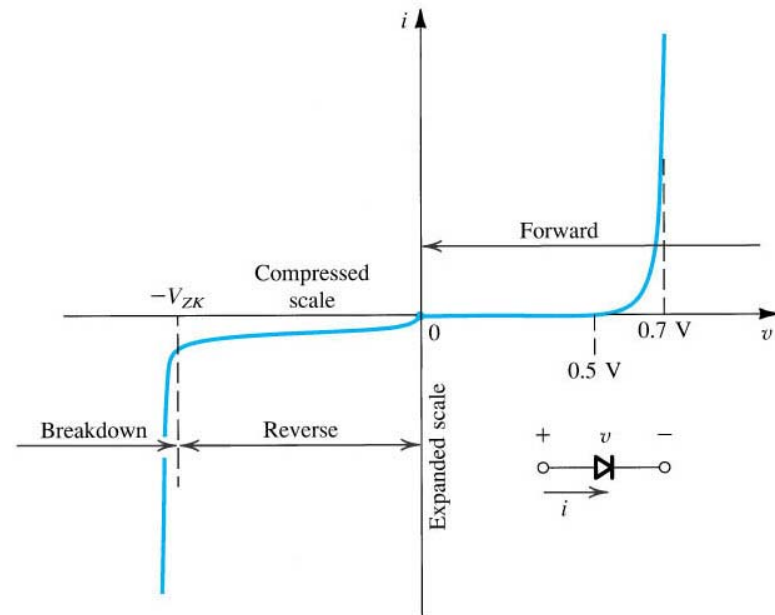
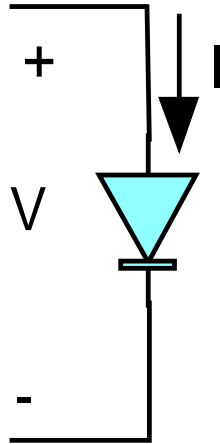
If trapezoidal method is applied

$$i_{n+1} = i_n + \frac{h}{2} [i'_{n+1} + i'_n]$$

$$v_{n+1} = \frac{2L}{h} i_{n+1} - \left(\frac{2L}{h} + v_n \right)$$



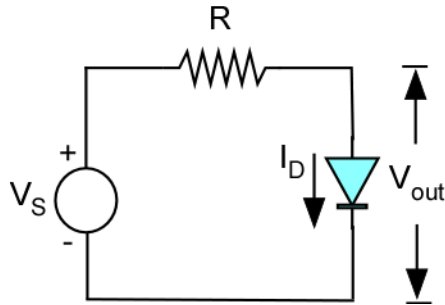
The Diode



- **Diode Properties**

- Two-terminal device that conducts current freely in one direction but blocks current flow in the opposite direction.
- The two electrodes are the anode which must be connected to a positive voltage with respect to the other terminal, the cathode in order for current to flow.

Diode Circuits

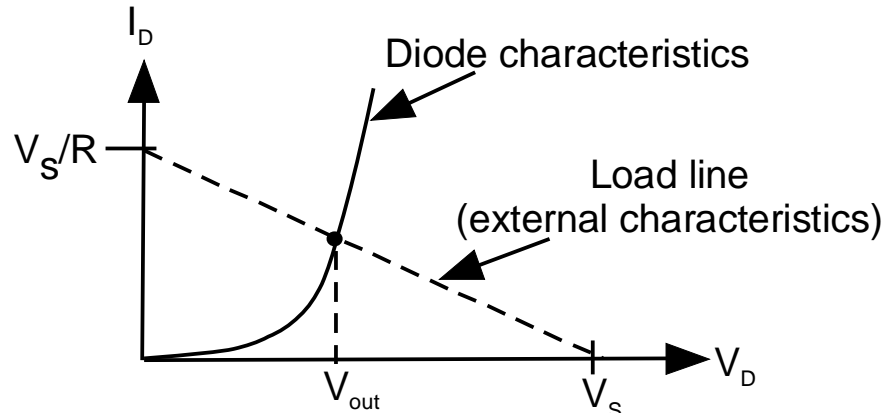


$$V_{out} = V_D$$

$$I_D = I_S \left(e^{V_D/V_T} - 1 \right)$$

$$V_S = RI_D + V_D = RI_D(V_D) + V_D$$

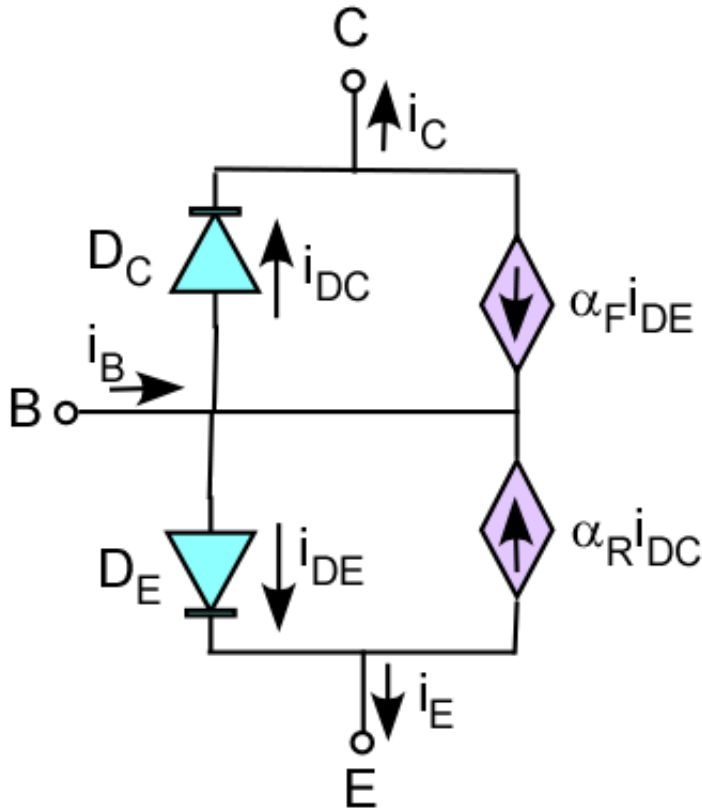
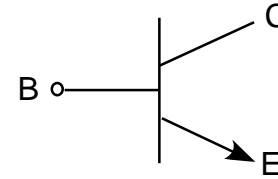
Nonlinear transcendental system → Use graphical method



Solution is found at intersection of load line characteristics and diode characteristics

BJT Ebers-Moll Model

NPN Transistor



$$i_E = \left(\frac{I_S}{\alpha_F} \right) \left(e^{v_{BE}/V_T} - 1 \right) - I_S \left(e^{v_{BC}/V_T} - 1 \right)$$

$$i_C = I_S \left(e^{v_{BE}/V_T} - 1 \right) - \left(\frac{I_S}{\alpha_R} \right) \left(e^{v_{BC}/V_T} - 1 \right)$$

$$i_B = \left(\frac{I_S}{\beta_F} \right) \left(e^{v_{BE}/V_T} - 1 \right) + \left(\frac{I_S}{\beta_R} \right) \left(e^{v_{BC}/V_T} - 1 \right)$$

$$\beta_F = \frac{\alpha_F}{1 - \alpha_F}$$

$$\beta_R = \frac{\alpha_R}{1 - \alpha_R}$$

Describes BJT operation in all of its possible modes

Newton Raphson Method

Problem: Wish to solve for $f(x)=0$

Use fixed point iteration method:

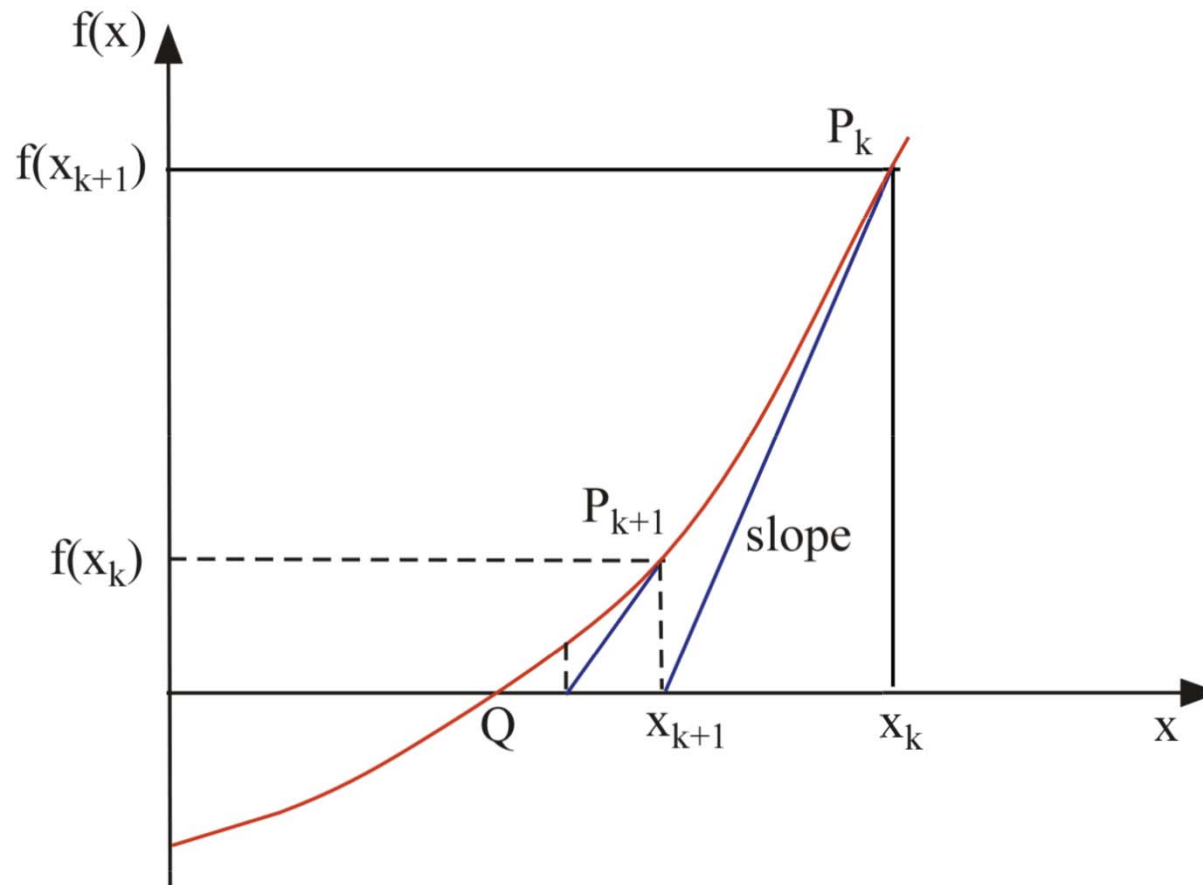
$$\text{Define } F(x) = x - K(x)f(x)$$

$$\mathfrak{J}: x_{k+1} = F(x_k) = x_k - K(x_k)f(x_k)$$

With Newton Raphson: $K(x) = [f'(x)]^{-1} = \left[\frac{df}{dx} \right]^{-1}$

therefore, $\mathfrak{J}: x_{k+1} = x_k - [f'(x_k)]^{-1} f(x_k)$

Newton Raphson Method (Graphical Interpretation)



Newton Raphson Algorithm

$$N - R: x_{k+1} = x_k - A_k^{-1} f(x_k)$$

$$A_k x_{k+1} = A_k x_k - f(x_k) \equiv S_k.$$

x_{k+1} is the solution of a linear system of equations.

$$A_k x = S_k \leftarrow \text{LU fact}$$

Forward and backward substitution.

A_k is the nodal matrix for N_k

S_k is the rhs source vector for N_k .

Newton Raphson Algorithm

0. $k \rightarrow 0$, gives \widehat{V}_0 ^{voltage controlled}, \widehat{i}_0 ^{current controlled}
1. Find V_k, i_k compute companion models.

$$\underbrace{G_k, I_k}_{V_c}, \underbrace{R_k, E_k}_{C_c}$$
2. Obtain A_k, S_k .
3. Solve $A_k x_k = S_k$.
4. $x_{k+1} \leftarrow$ Solution
5. Check for convergence $\|x_{k+1} - x_k\| < \varepsilon$.
 If they converge, then stop.
6. $k + 1 \rightarrow k$, and go to step 1.

Newton Raphson - Diode

It is obvious from the circuit that the solution must satisfy

$$f(V) = 0$$

We also have

$$f'(V) = \frac{1}{R} + \frac{I_s}{V_t} e^{V/V_t}$$

The Newton method relates the solution at the (k+1)th step to the solution at the kth step by

$$V_{k+1} = -\frac{f(V_k)}{f'(V_k)} + V_k$$

$$V_{k+1} = V_k - \frac{\frac{V_k - E}{R} + I_s (e^{V_k/V_t} - 1)}{\frac{1}{R} + \frac{I_s}{V_t} e^{V_k/V_t}}$$

Newton Raphson

After manipulation we obtain

$$\left(\frac{1}{R} + g_k \right) V_{k+1} = \frac{E}{R} - J_k$$

$$g_k = \frac{I_s}{V_t} e^{V_k/V_t}$$

$$J_k = I_s (e^{V_k/V_t} - 1) - V_k g_k$$

Diode Circuit – Iterative Method

Newton-Raphson Method

$$\text{Use: } x_{k+1} = x_k - [f'(x_k)]^{-1} f(x_k)$$

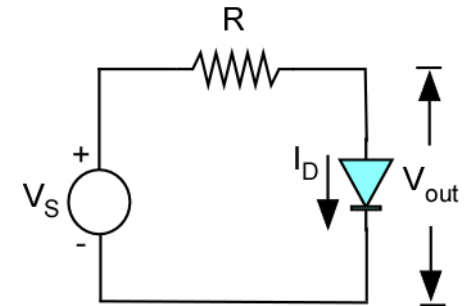
$$x^{(k+1)} = x^{(k)} - [f'(x^{(k)})]^{-1} f(x^{(k)}) \quad V_{out} = V_D$$

$$f(V_D) = \frac{V_D - V_S}{R} + I_S (e^{V_D/V_T} - 1) = 0$$
$$f'(V_D) = \frac{1}{R} + \frac{I_S}{V_T} e^{V_D/V_T}$$

$$V_D^{(k+1)} = V_D^{(k)} - \frac{\frac{V_D^{(k)} - V_S}{R} + I_S (e^{V_D^{(k)}/V_T} - 1)}{\frac{1}{R} + \frac{I_S}{V_T} e^{V_D^{(k)}/V_T}}$$

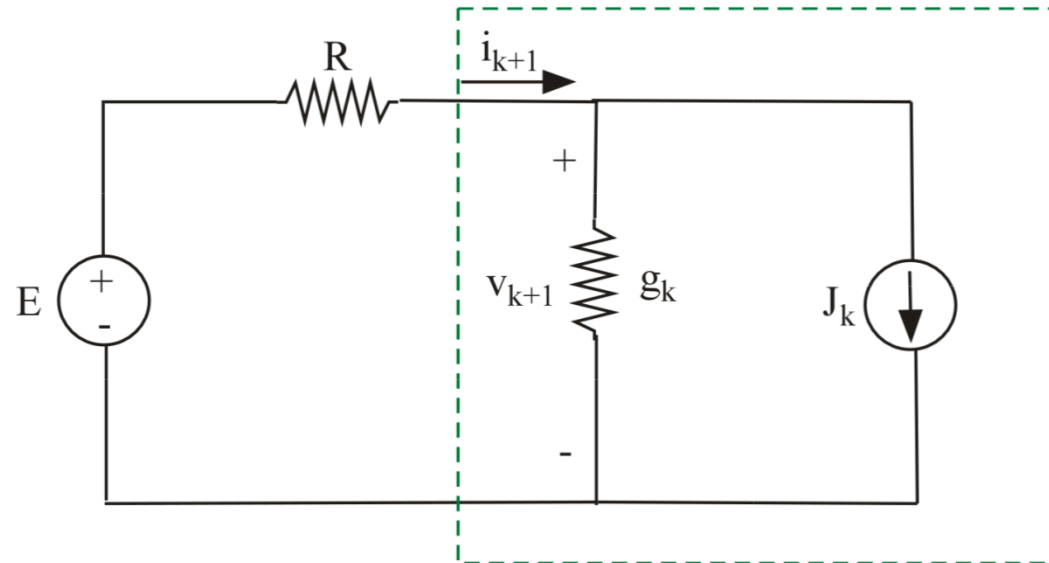
Where $V_D^{(k)}$ is the value of V_D at the k th iteration

Procedure is repeated until convergence to final (true) value of V_D which is the solution. Rate of convergence is quadratic.



Newton Raphson for Diode

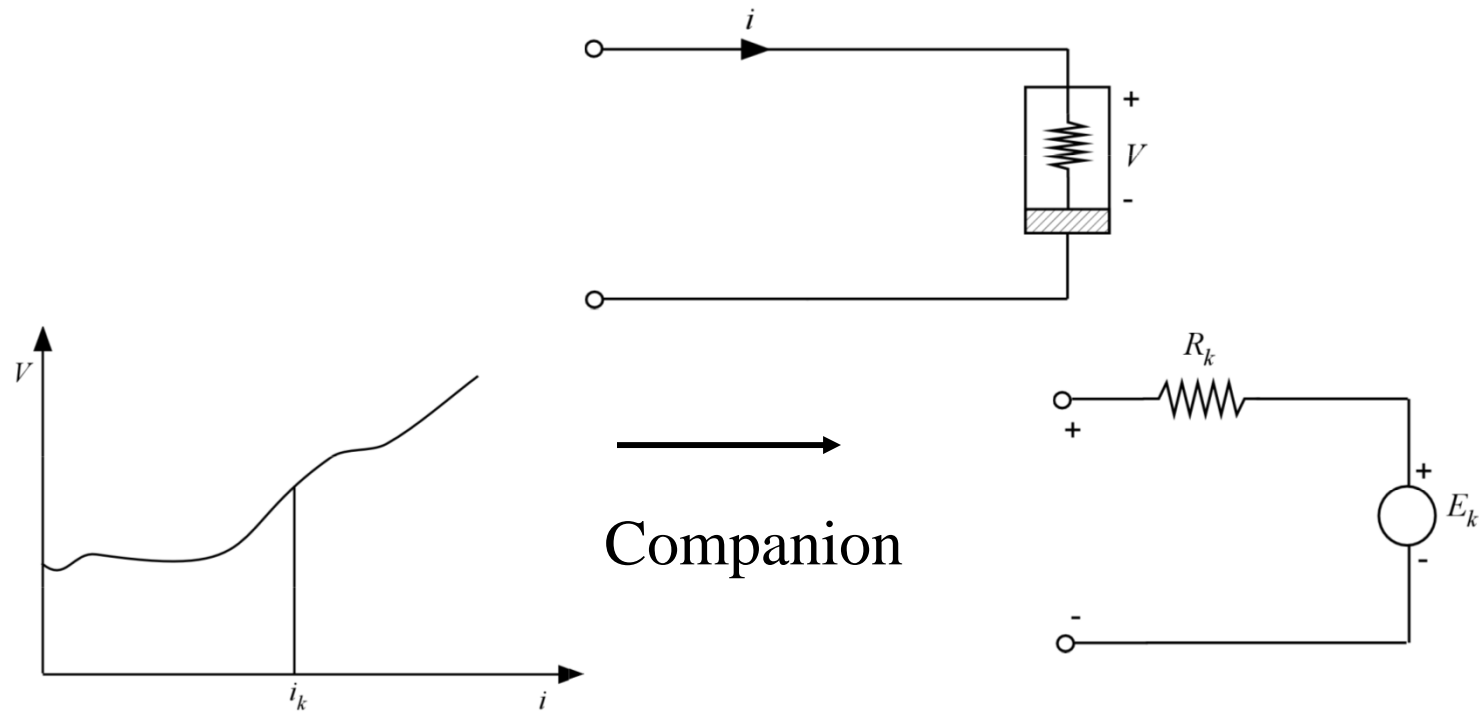
Newton-Raphson representation of diode circuit at kth iteration



$$g_k = \frac{I_s}{V_t} e^{V_k/V_t}$$

$$J_k = I_s (e^{V_k/V_t} - 1) - V_k g_k$$

Current Controlled



$$R_k = \left. \frac{dh(i)}{di} \right|_{i=i_k}$$

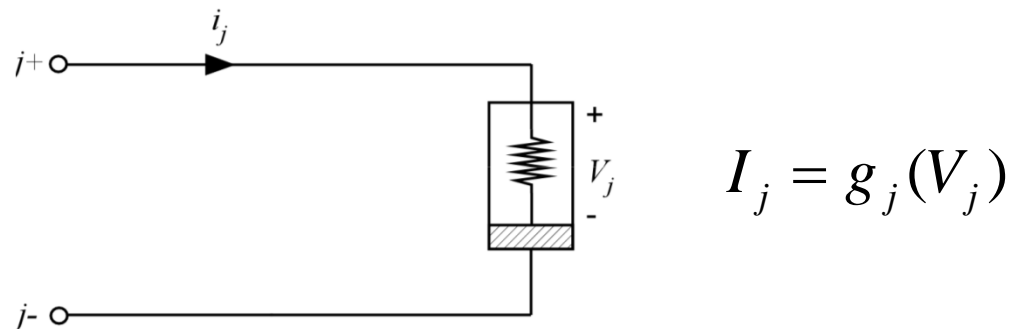
$$E_k = h(i_k) - R_k i_k$$

General Network

Let x = vector variables in the network to be solved for. Let $f(x) = 0$ be the network equations. Let x_k be the present iterate, and define

$$A_k = f'(x_k) \rightarrow \text{Jacobian of } f \text{ at } x = x_k$$

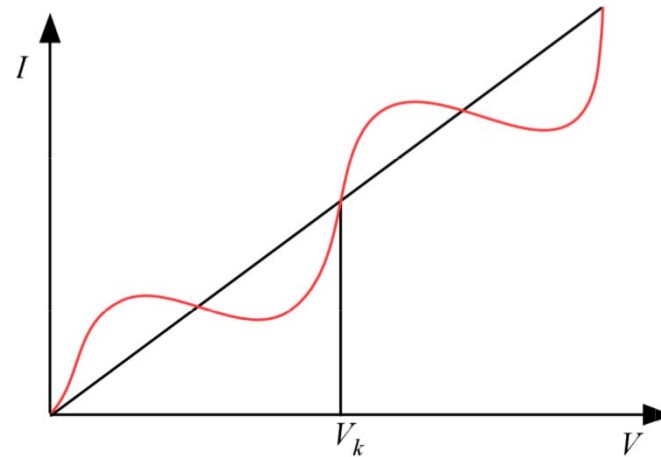
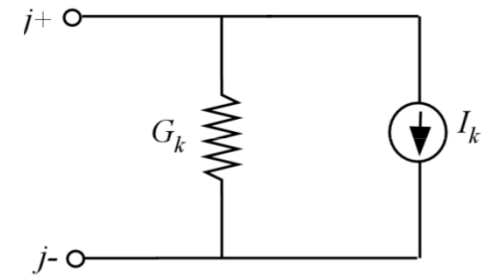
Let N_k be the linear network where each non-linear resistor is replaced by its companion model computed from x_k .



General Network

$$V_{jk} = P_{j+k} - P_{j-k}$$

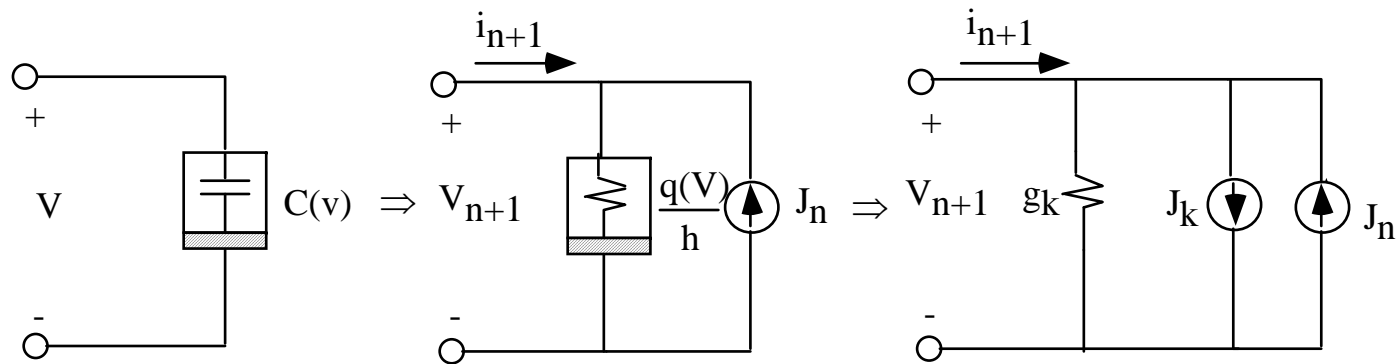
Companion
model



$$G_k = \left. \frac{dg(V)}{dV} \right|_{V=V_k}$$

$$I_k = g[V_k] - G_k V_k$$

Nonlinear Reactive Elements



$$q = f(v), \quad i = \frac{dq}{dt}$$

$$q_{n+1} = q_n + h \left. \frac{dq}{dt} \right|_{t=t_{n+1}}$$

$$\text{or, } i_{n+1} = \frac{q_{n+1} - q_n}{h} \Rightarrow i_{n+1}(v_{n+1}) = \frac{f(v_{n+1})}{h}$$

General Element

