ECE 546
Lecture -16
MNA and SPICE

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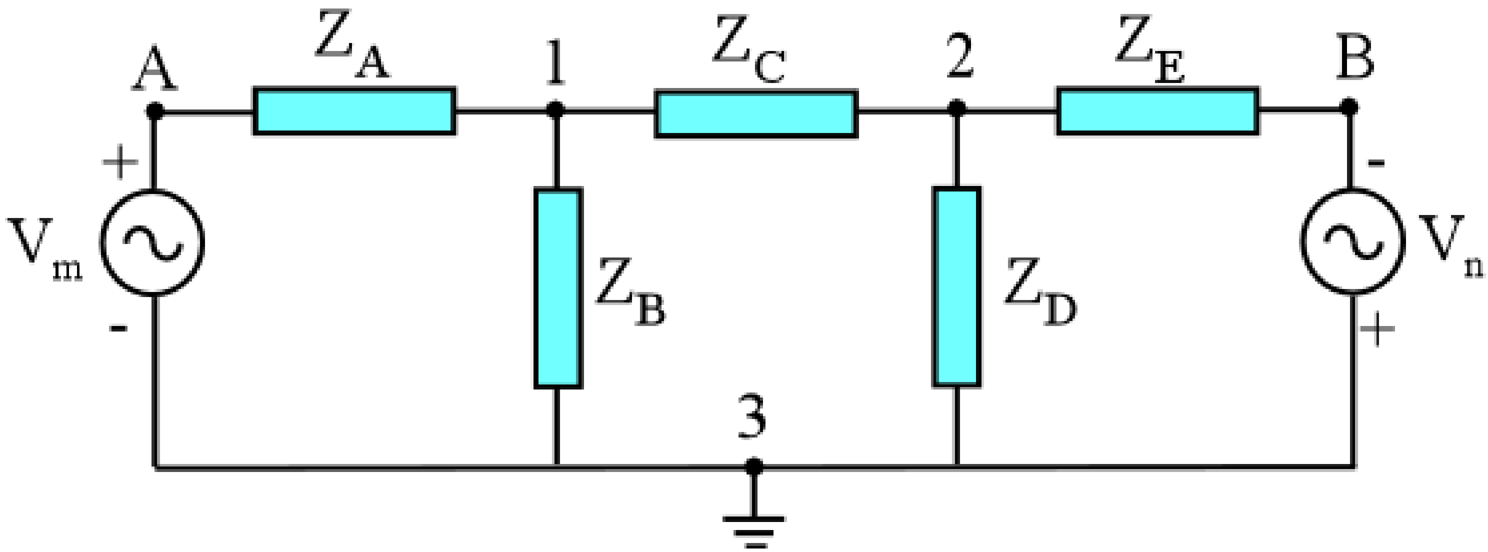
Nodal Analysis

The Node Voltage method consists in determining potential differences between nodes and ground (reference) using KCL



For Node 1: $\frac{V_1 - V_m}{Z_A} + \frac{V_1}{Z_B} + \frac{V_1 - V_2}{Z_C} = 0$

Nodal Analysis

For Node 2:

$$\frac{V_2 - V_1}{Z_C} + \frac{V_2}{Z_D} + \frac{V_2 - V_n}{Z_E} = 0$$

Rearranging the terms gives:

$$\left(\frac{1}{Z_A} + \frac{1}{Z_B} + \frac{1}{Z_C}\right)V_1 - \left(\frac{1}{Z_C}\right)V_2 = \left(\frac{1}{Z_A}\right)V_m$$

$$-\left(\frac{1}{Z_C}\right)V_1 + \left(\frac{1}{Z_C} + \frac{1}{Z_D} + \frac{1}{Z_E}\right)V_2 = -\left(\frac{1}{Z_E}\right)V_n$$

Defining:

$$G_A = \frac{1}{Z_A},\ G_B = \frac{1}{Z_B},\ G_C = \frac{1}{Z_C},\ G_D = \frac{1}{Z_D},\ G_E = \frac{1}{Z_E}$$

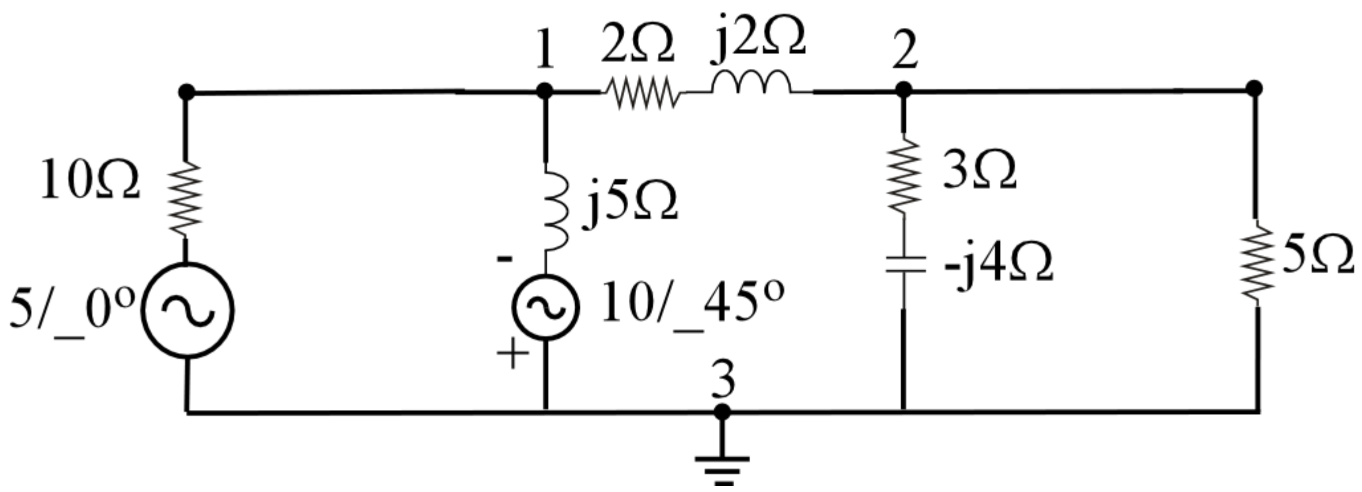
Nodal Analysis

Rearranging the terms gives:

$$\begin{bmatrix} (G_A + G_B + G_C) & -G_C \\ -G_C & (G_C + G_D + G_E) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} G_A V_m \\ -G_E V_m \end{bmatrix}$$

The system can be solved to yield V_1 and V_2.

Nodal Analysis



For Node 1:

$$\frac{V_1 - 5\angle 0^\circ}{10} + \frac{V_1 + j10\angle 45^\circ}{j5} + \frac{V_1 - V_2}{2 + j2} = 0$$

For Node 2:

$$\frac{V_2 - V_1}{2 + j2} + \frac{V_2}{3 - j4} + \frac{V_2}{5} = 0$$

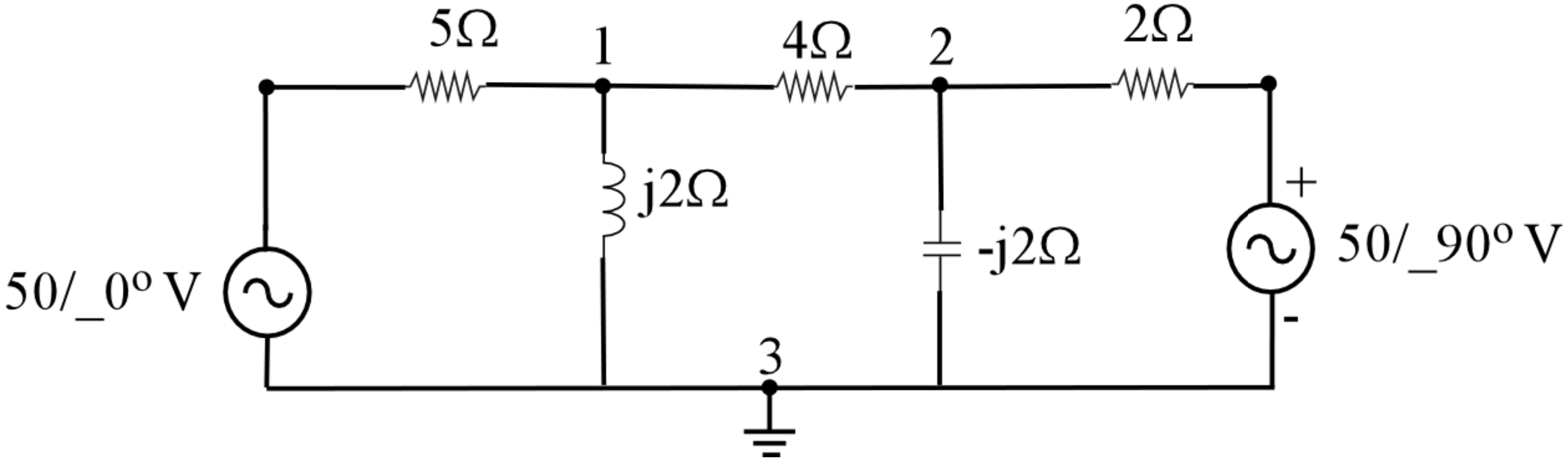
Nodal Analysis

Rearranging the terms gives:

$$\left(\frac{1}{10}+\frac{1}{j5}+\frac{1}{2+j2}\right)V_1-\left(\frac{1}{2+j2}\right)V_2=\frac{5\angle 0^\circ}{10}-\frac{10\angle 45^\circ}{j5}$$

$$-\left(\frac{1}{2+j2}\right)V_1+\left(\frac{1}{2+j2}+\frac{1}{3-j4}+\frac{1}{5}\right)V_2=0$$

Nodal Analysis



$$\begin{bmatrix} \left(\frac{1}{5}+\frac{1}{j2}+\frac{1}{4}\right) & -\left(\frac{1}{4}\right) \\ -\left(\frac{1}{4}\right) & \left(\frac{1}{4}+\frac{1}{-j2}+\frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{50\angle 0^\circ}{5} \\ \frac{50\angle 90^\circ}{5} \end{bmatrix}$$

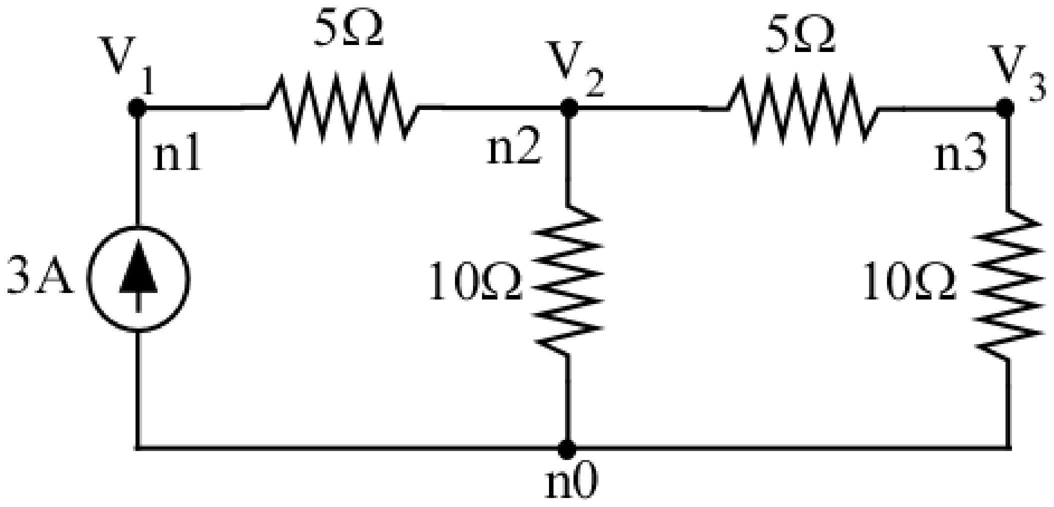
[Y] [v]=[i]

Nodal Analysis - Solution

$$V_1 = \frac{\begin{vmatrix} 10 & 0.25 \\ j25 & 0.75 + j0.5 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ 0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{13.5\angle 56.3^\circ}{0.546\angle -15.95^\circ} = 24.7\angle 72.25^\circ\ V$$

$$V_2 = \frac{\begin{vmatrix} 0.45 - j0.5 & 10 \\ -0.25 & j25 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ 0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{18.35\angle 37.8^\circ}{0.546\angle -15.95^\circ} = 33.6\angle 53.75^\circ\ V$$

Nodal Formulation



Node 1: $-3+\dfrac{V_1-V_2}{5}=0$

Node 2: $\dfrac{V_2-V_1}{5}+\dfrac{V_2}{10}+\dfrac{V_2-V_3}{5}=0$

Node 3: $\dfrac{V_3-V_2}{5}+\dfrac{V_3}{10}=0$

Nodal Solution

Arrange in matrix form: [G][V]=[I]

$$\begin{bmatrix} 0.2 & -0.2 & 0 \\ -0.2 & 0.5 & -0.2 \\ 0 & -0.2 & 0.3 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Use Gaussian elimination to form an upper triangular matrix

$$\begin{bmatrix} 0.2 & -0.2 & 0 \\ 0 & 0.3 & -0.2 \\ 0 & 0 & 0.25 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

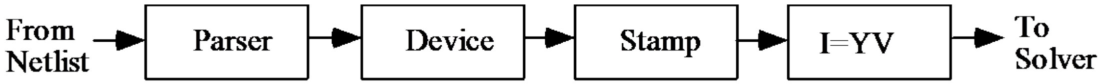
Solve for V_1, V_2 and V_3 using backward substitution

This can always be solved no matter how large the matrix is

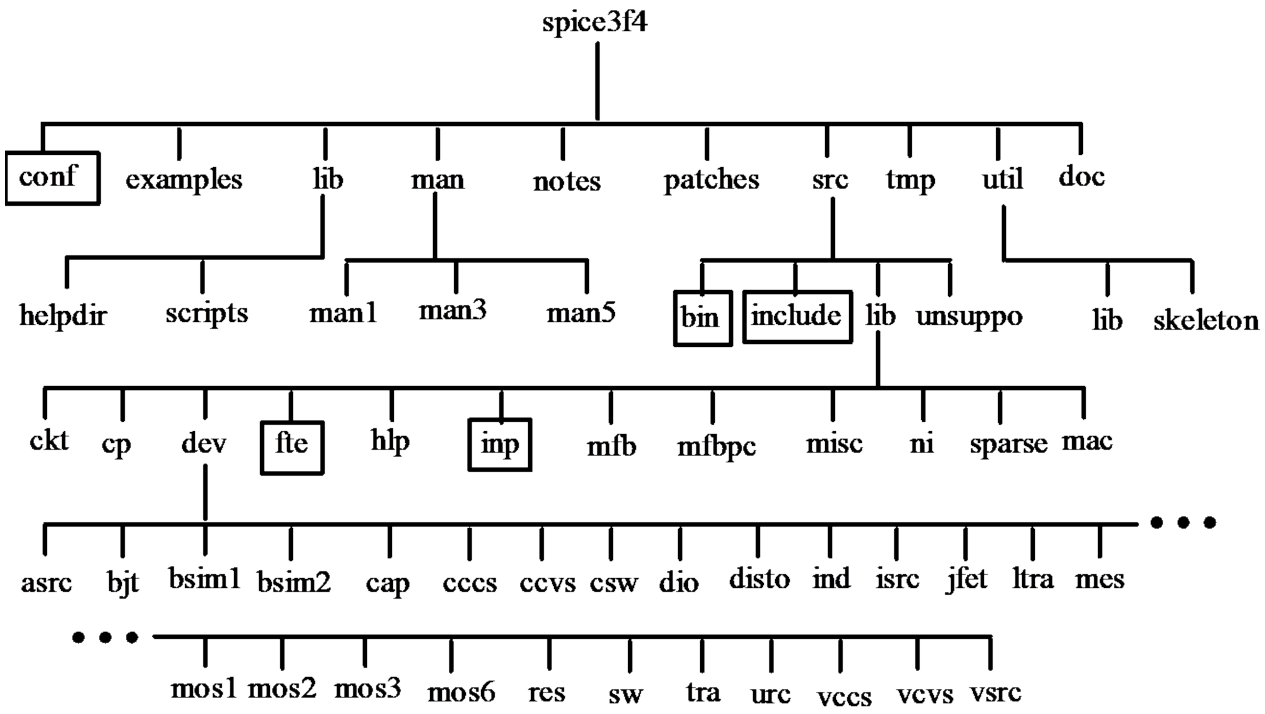
Why SPICE ?

- Established platform
- Powerful engine
- Source code available for free
- Extensive libraries of devices
- New device installation procedure easy

SPICE



SPICE Directory Structure



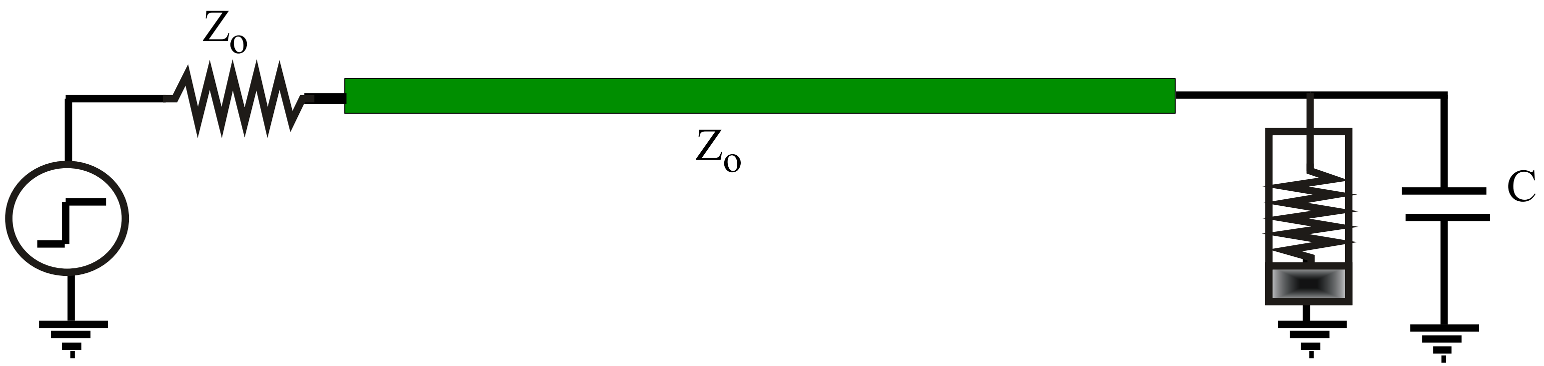
MOS SPICE Parameters

Symbol	Description	Value	Units
L_{drawn}	Device length (drawn)	0.35	μm
L_{eff}	Device length (effective)	0.25	μm
t_{ox}	Gate oxide thickness	70	A
N_a	Density of acceptor ions in NFET channel	1.0×10^{17}	cm^{-3}
N_d	Density of donor ions in PFET channel	2.5×10^{17}	cm^{-3}
V_{Tn}	NFET threshold voltage	0.5	V
V_{Tp}	PFET threshold voltage	-0.5	V
λ	Channel modulation parameter	0.1	V^{-1}
γ	Body effect parameter	0.3	$V^{1/2}$
V_{sat}	Saturation velocity	1.7×10^5	m/s
μ_n	Electron mobility	400	cm^2/Vs
μ_p	Hole mobility	100	cm^2/Vs
k_n	NFET process transconductance	200	$\mu A/V^2$
k_p	PFET process transconductance	50	$\mu A/V^2$
C_{ox}	Gate oxide capacitance per unit area	5	$fF/\mu m^2$
C_{GSO}, C_{GDO}	Gate source and drain overlap capacitance	0.1	fF/μm
C_J	Junction capacitance	0.5	$fF/\mu m^2$
C_{JSW}	Junction sidewall capacitance	0.2	fF/μm
R_{poly}	Gate sheet resistance	4	Ω/square
R_{diff}	Source and drain sheet resistance	4	Ω/square

Problems

- **Nonlinear Devices**
  - Diodes, transistors cannot be simulated in the frequency domain
  - Capacitors and inductors are best described in the frequency domain
  - Use time-domain representation for reactive elements (capacitors and inductors)
- **Circuit Size**
  - Matrix size becomes prohibitively large

Motivations



- Loads are nonlinear
- Need to model reactive elements in the time domain
- Generalize to nonlinear reactive elements

Time-Domain Model for Linear Capacitor

For linear capacitor C with voltage v and current i which must satisfy

$$i = C \frac{dv}{dt}$$

Using the backward Euler scheme, we discretize time and voltage variables and obtain at time $t = nh$

$$v_{n+1} = v_n + hv'_{n+1}$$

Time-Domain Model for Linear Capacitor

After substitution, we obtain

$$v'_{n+1} = \frac{i_{n+1}}{C}$$

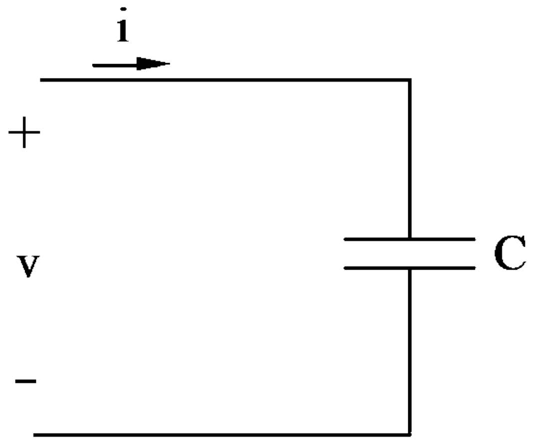
so that

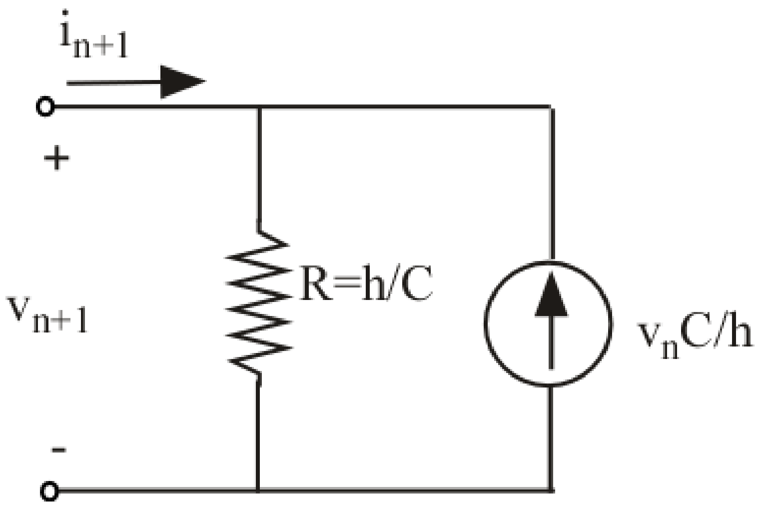
$$v_{n+1} = v_n + h\ \frac{i_{n+1}}{C}$$

The solution for the current at t_{n+1} is, therefore,

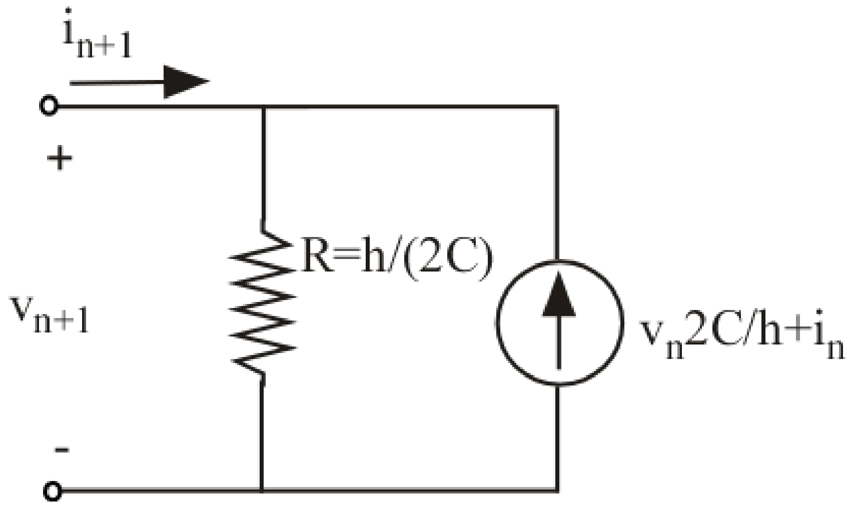
$$i_{n+1} = \frac{C}{h} v_{n+1} - \frac{C}{h} v_n$$

Time-Domain Model for Linear Capacitor





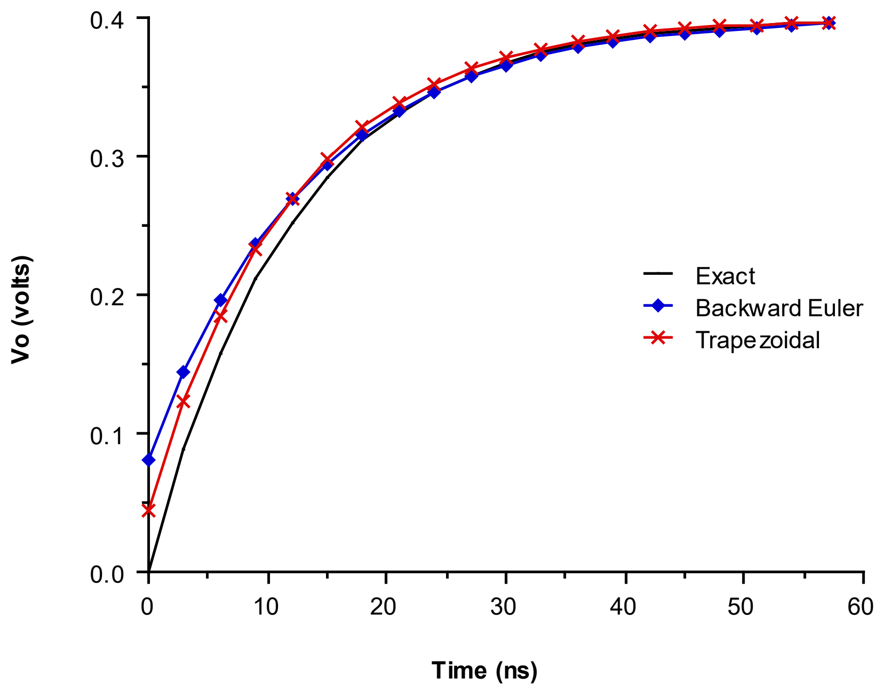
Backward Euler companion model at t=nh



Trapezoidal companion model at t=nh

Time-Domain Model for Linear Capacitor

Step response comparisons



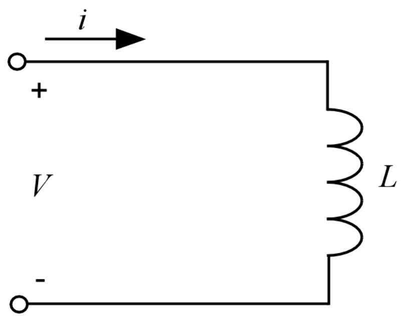
Time-Domain Model for Linear Inductor

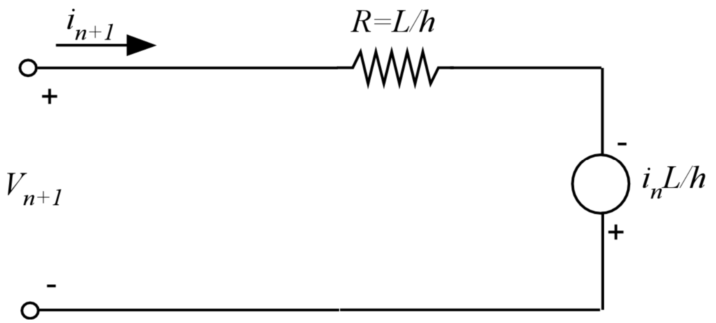
$$v = L\frac{di}{dt}$$

$$\text{Backward Euler}: \quad i_{n+1} = i_n + h\, i'_{n+1}$$

$$i'_{n+1} = \frac{v_{n+1}}{L}$$

$$v_{n+1} = \frac{L}{h} i_{n+1} - \frac{L}{h} i_n$$



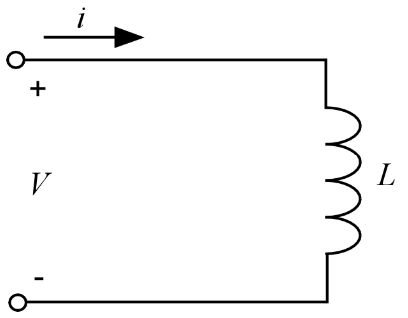


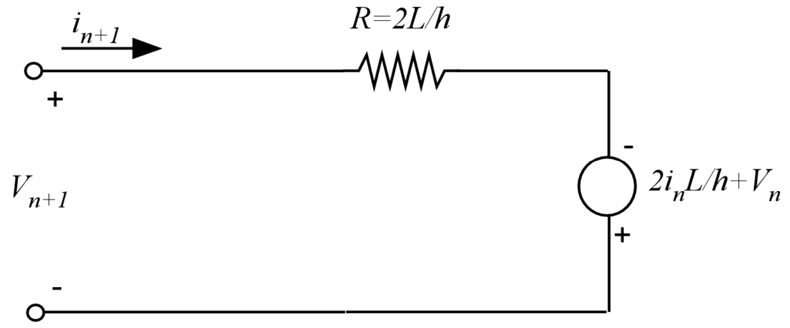
Time-Domain Model for Linear Inductor

If trapezoidal method is applied

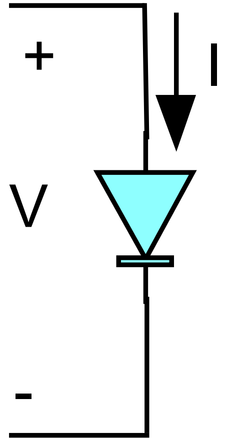
$$\frac{v_{n+1} + v_n}{2} = L\frac{i_{n+1} - i_n}{h}$$

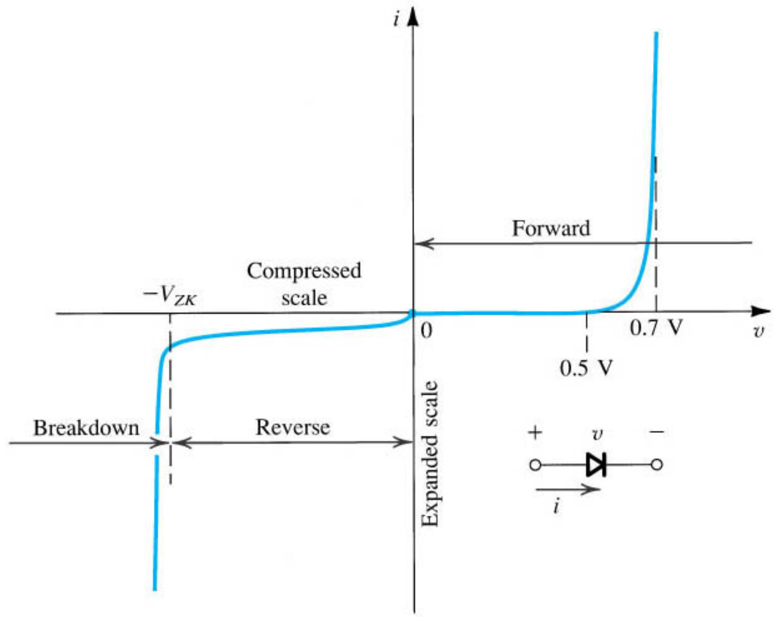
$$v_{n+1} = \frac{2L}{h} i_{n+1} - \left(\frac{2L}{h} i_n + v_n \right)$$





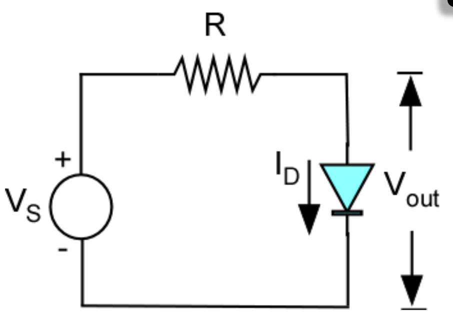
The Diode





- **Diode Properties**
  - Two-terminal device that conducts current freely in one direction but blocks current flow in the opposite direction.
  - The two electrodes are the anode which must be connected to a positive voltage with respect to the other terminal, the cathode in order for current to flow.

Diode Circuits

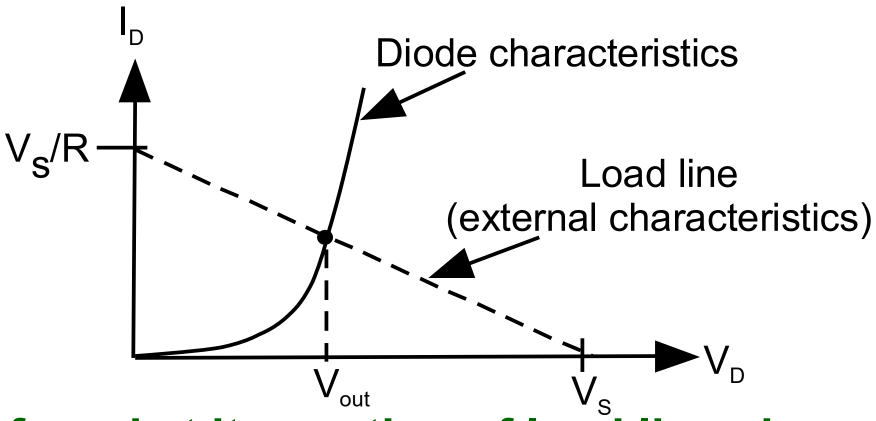


$$V_{out} = V_D$$

$$I_D = I_S\left(e^{V_D/V_T} - 1\right)$$

$$V_S = RI_D + V_D = RI_D(V_D) + V_D$$

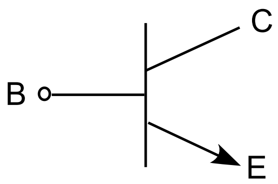
Nonlinear transcendental system ➔ Use graphical method

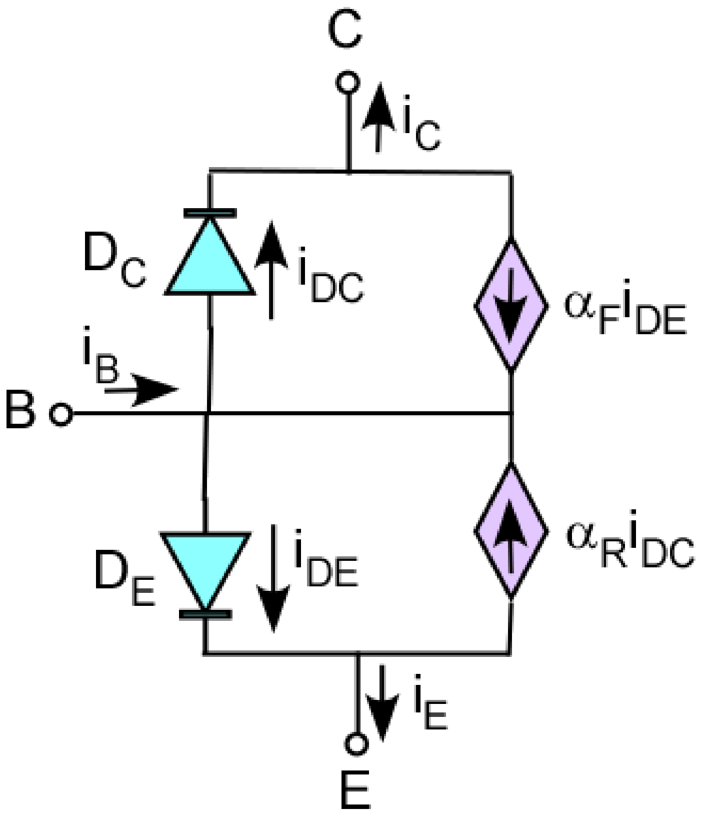


Solution is found at itersection of load line characteristics and diode characteristics

BJT Ebers-Moll Model

NPN Transistor





$$i_E = \left(\frac{I_S}{\alpha_F}\right)\left(e^{v_{BE}/V_T} - 1\right) - I_S\left(e^{v_{BC}/V_T} - 1\right)$$

$$i_C = I_S\left(e^{v_{BE}/V_T} - 1\right) - \left(\frac{I_S}{\alpha_R}\right)\left(e^{v_{BC}/V_T} - 1\right)$$

$$i_B = \left(\frac{I_S}{\beta_F}\right)\left(e^{v_{BE}/V_T} - 1\right) + \left(\frac{I_S}{\beta_R}\right)\left(e^{v_{BC}/V_T} - 1\right)$$

$$\beta_F = \frac{\alpha_F}{1-\alpha_F} \qquad \beta_R = \frac{\alpha_R}{1-\alpha_R}$$

Describes BJT operation in all of its possible modes

Newton Raphson Method

<u>Problem:</u> Wish to solve for f(x)=0

Use fixed point iteration method:

$$Define\ \ F(x) = x - K(x)f(x)$$

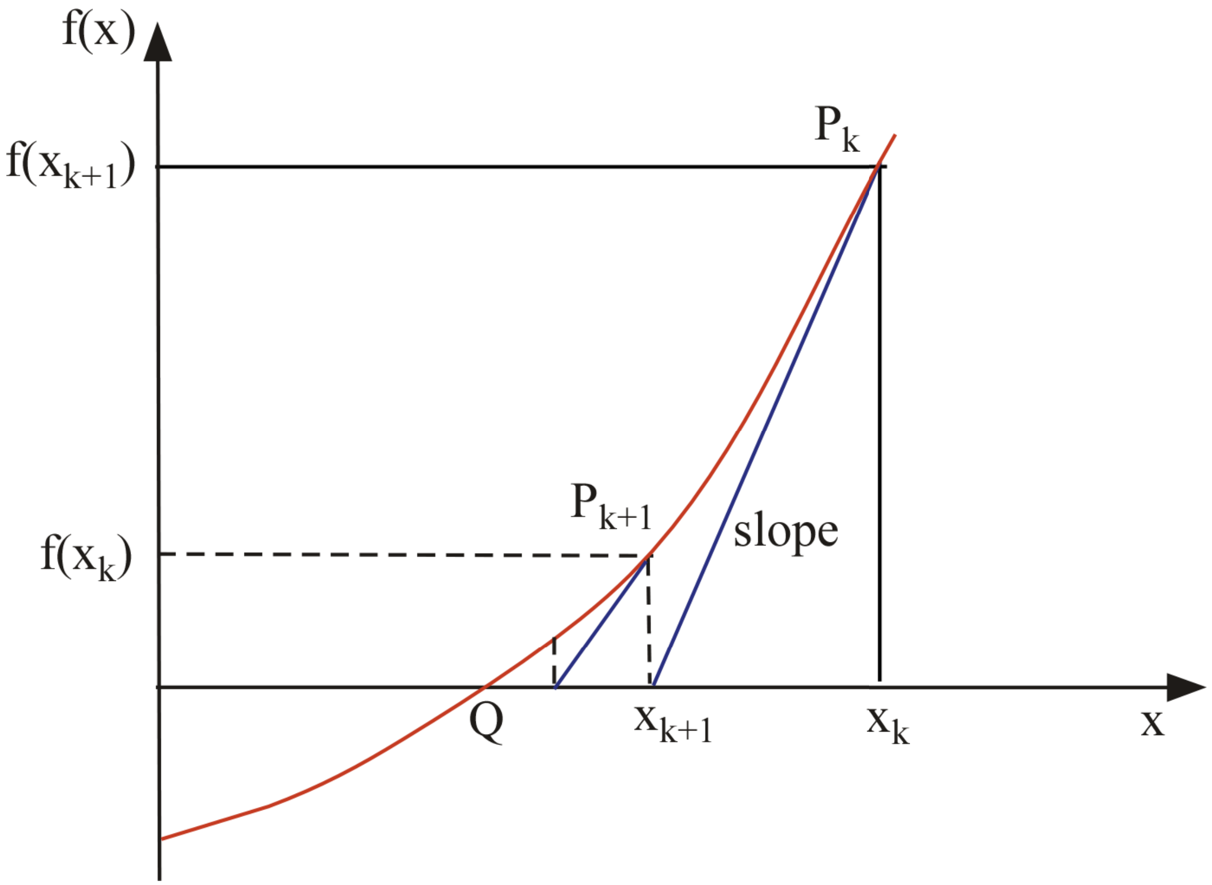
$$\Im: \quad x_{k+1} = F(x_k) = x_k - K(x_k)f(x_k)$$

With Newton Raphson: $K(x) = [f'(x)]^{-1} = \left[\frac{df}{dx}\right]^{-1}$

therefore, $\Im: \ x_{k+1} = x_k - [f'(x_k)]^{-1} f(x_k)$

Newton Raphson Method

(Graphical Interpretation)



Newton Raphson Algorithm

$$N - R: \ x_{k+1} = x_k - A_k^{-1} f(x_k)$$

$$A_k x_{k+1} = A_k x_k - f(x_k) \equiv S_k .$$

x_{k+1} is the solution of a linear system of equations.

$$A_k x = S_k \quad \leftarrow \text{ LU fact}$$

Forward and backward substitution.

A_k is the nodal matrix for N_k

S_k is the rhs source vector for N_k.

Newton Raphson Algorithm

$$0.\ \ k \rightarrow 0,\ gives\ \ \overbrace{V_0}^{voltage\ controlled}\ ,\ \overbrace{i_0}^{current\ controlled}$$

$$1.\ \ Find\ \ V_k,\ i_k\ \ compute\ \ companion\ \ \text{mod}els.$$

$$\underbrace{G_k,\ I_k}_{V_c},\ \underbrace{R_k,\ E_k}_{C_c}$$

$$2.\ \ Obtain\ \ A_k,\ S_k.$$

$$3.\ \ Solve\ \ A_k x_k = S_k.$$

$$4.\ \ x_{k+1} \leftarrow Solution$$

$$5.\ \ Check\ \ for\ \ convergence\ \left\| x_{k+1} - x_k \right\| < \varepsilon.$$

$$If\ \ they\ converge,\ then\ stop.$$

$$6.\ \ k+1 \rightarrow k,\ and\ \ go\ to\ step\ 1.$$

Newton Raphson - Diode

It is obvious from the circuit that the solution must satisfy

f(V) = 0

We also have

$$f'(V) = \frac{1}{R} + \frac{I_s}{V_t} e^{V/V_t}$$

The Newton method relates the solution at the (k+1)th step to the solution at the kth step by

$$V_{k+1} = -\frac{f(V_k)}{f'(V_k)} + V_k$$

$$V_{k+1} = V_k - \frac{\frac{V_k - E}{R} + I_s\left(e^{V_k/V_t} - 1\right)}{\frac{1}{R} + \frac{I_s}{V_t} e^{V_k/V_t}}$$

Newton Raphson

After manipulation we obtain

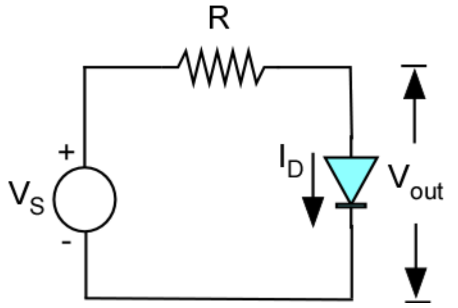
$$\left(\frac{1}{R}+g_k\right)V_{k+1}=\frac{E}{R}-J_k$$

$$g_k=\frac{I_s}{V_t}e^{V_k/V_t}$$

$$J_k=I_s\left(e^{V_k/V_t}-1\right)-V_k g_k$$

Diode Circuit – Iterative Method

Newton-Raphson Method



Use: $x_{k+1} = x_k - \left[f'(x_k) \right]^{-1} f(x_k)$

$$x^{(k+1)} = x^{(k)} - \left[f'(x^{(k)}) \right]^{-1} f(x^{(k)})$$

$$V_{out} = V_D$$

$$f(V_D) = \frac{V_D - V_S}{R} + I_S\left(e^{V_D/V_T} - 1\right) = 0$$

$$f'(V_D) = \frac{1}{R} + \frac{I_S}{V_T} e^{V_D/V_T}$$

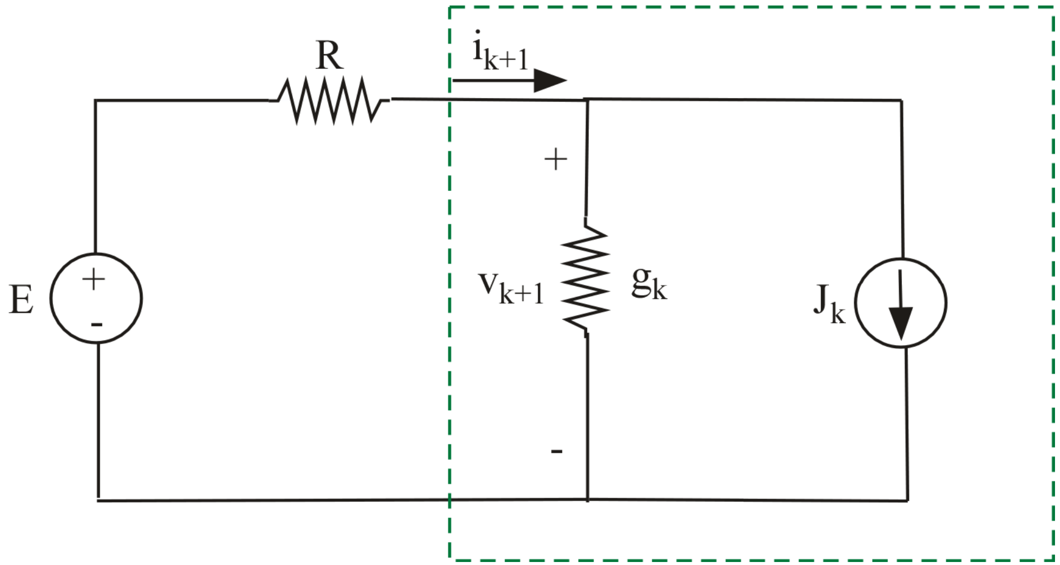
$$V_D^{(k+1)} = V_D^{(k)} - \frac{\dfrac{V_D^{(k)} - V_S}{R} + I_S\left(e^{V_D^{(k)}/V_T} - 1\right)}{\dfrac{1}{R} + \dfrac{I_S}{V_T} e^{V_D^{(k)}/V_T}}$$

Where $V_D^{(k)}$ is the value of V_D at the kth iteration

Procedure is repeated until convergence to final (true) value of V_D which is the solution. Rate of convergence is quadratic.

Newton Raphson for Diode

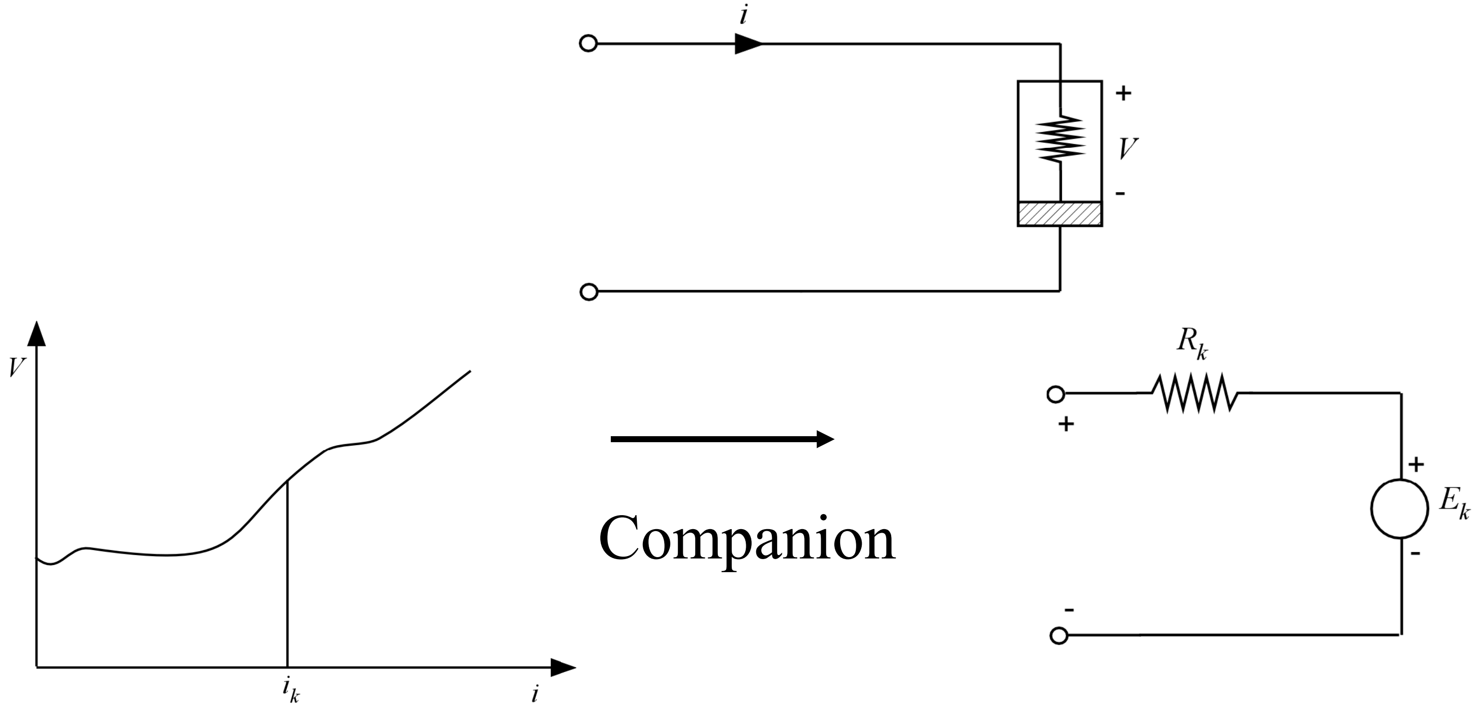
Newton-Raphson representation of diode circuit at kth iteration



$$g_k = \frac{I_s}{V_t} e^{V_k / V_t}$$

$$J_k = I_s\left(e^{Vk/Vt} - 1\right) - V_k g_k$$

Current Controlled



$$R_k = \left. \frac{dh(i)}{di} \right|_{i=i_k}$$

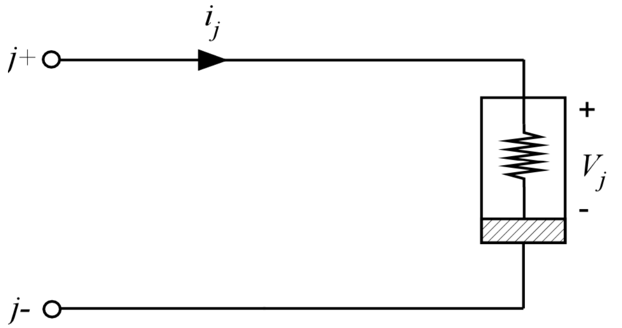
$$E_k = h(i_k) - R_k i_k$$

General Network

Let x = vector variables in the network to be solved for. Let $f(x) = 0$ be the network equations. Let x_k be the present iterate, and define

$$A_k = f'(x_k) \rightarrow Jacobian\ of\ \ f\ \ at\ \ x = x_k$$

Let N_k be the linear network where each non-linear resistor is replaced by its companion model computed from x_k.

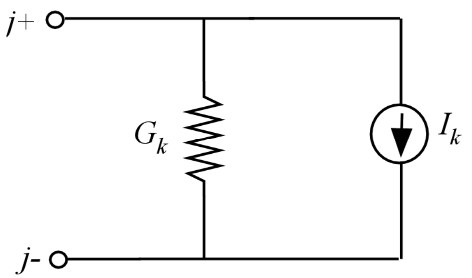


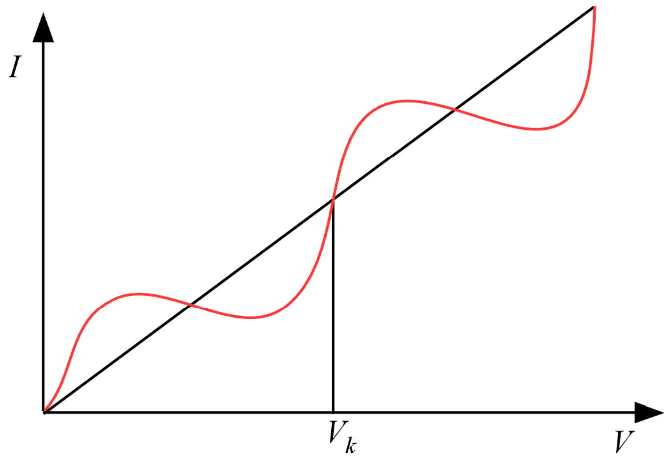
$$I_j = g_j(V_j)$$

General Network

$$V_{jk} = P_{j+k} - P_{j-k}$$

Companion model →

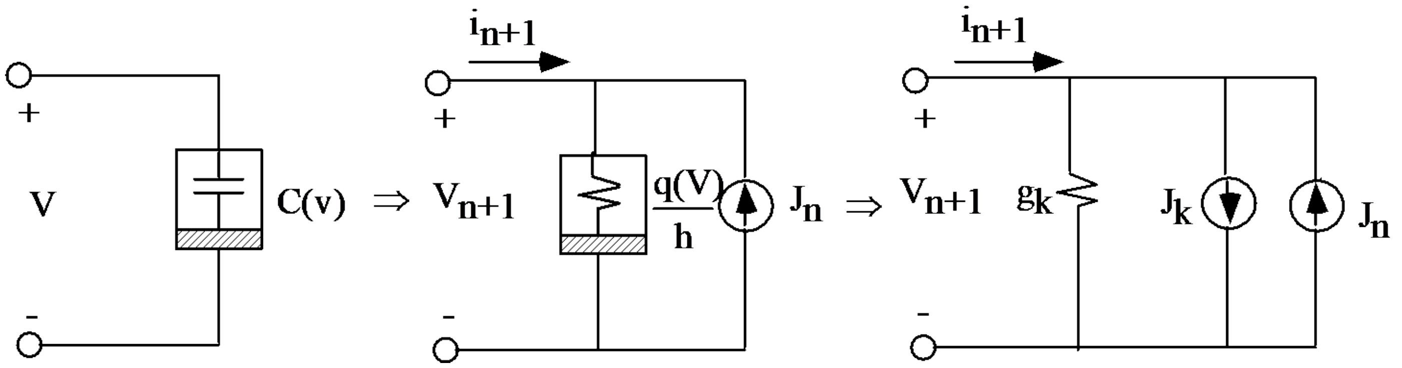




$$G_k = \left.\frac{dg(V)}{dV}\right|_{V=V_k}$$

$$I_k = g\left[V_k\right] - G_k V_k$$

Nonlinear Reactive Elements

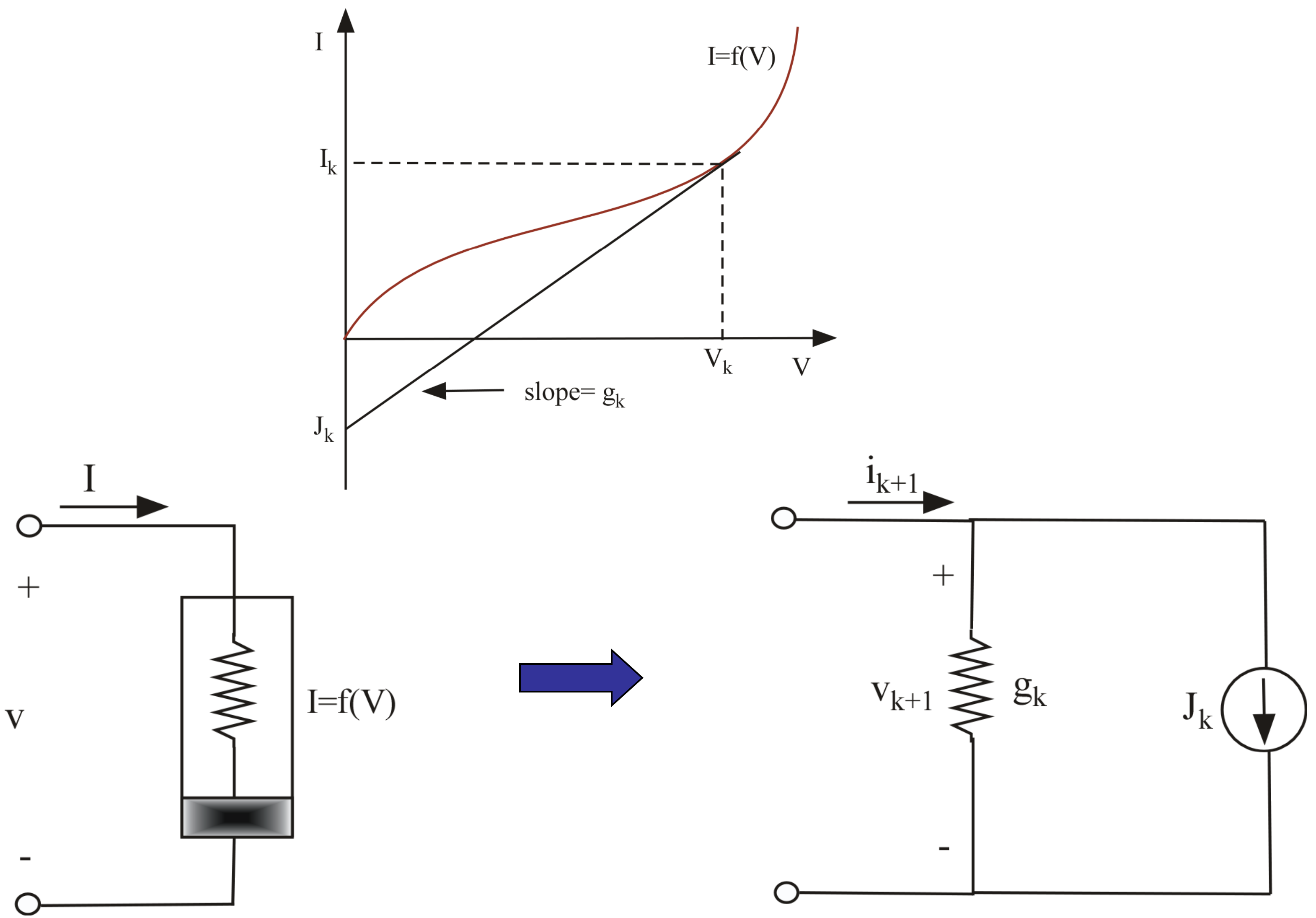


$$q = f(v)\ ,\quad i = \frac{dq}{dt}$$

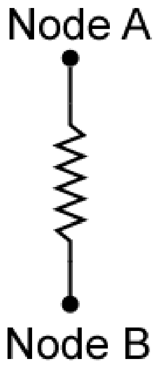
$$q_{n+1} = q_n + h\left.\frac{dq}{dt}\right|_{t=t_{n+1}}$$

$$or,\quad i_{n+1} = \frac{q_{n+1}}{h} - \frac{q_n}{h} \Rightarrow i_{n+1}(v_{n+1}) = \frac{f(v_{n+1})}{h}$$

General Element



Stamp in SPICE



Column For Node A — Column For Node B

Row For Node A → / Row For Node B →

$$\underbrace{\begin{bmatrix} +G_R & -G_R \\ -G_R & +G_R \end{bmatrix}}_{\text{Conductance array}} \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{\text{Voltage array}} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{\text{Current array}}$$

SPICE solves

[Y] [v]=[i]

$$\underbrace{\begin{bmatrix} \cdot & & & \\ & [G] & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix}}_{\mathbf{Y}} \underbrace{\begin{bmatrix} [\ ] \\ \\ \\ \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} [\ ] \\ \\ \\ \end{bmatrix}}_{\mathbf{i}}$$