Proceedings Letters

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Contributions should be submitted in triplicate to the Editor, PROCEEDINGS OF THE IEEE, *345 East 47 Street, New York, N. Y. 10017. The length should be limited to five double-spaced typewritten pages, counting each illustration as half a page. An abstract of 50 words or less and the original figures should be included. Instructions covering abbreviations, the form for references, general style, and the preparation of figures are found in "Information for IEEE Authors," available on request from the IEEE Editorial Department. Authors are invited to suggest the categories in the table of contents under which their letters best fit.*

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The Discrete Hilbert Transform

Abstract—A Hilbert transformation procedure for discrete data has been developed. This transform could be useful in a variety of applications such as the analysis of sampled data systems and the simulation of filters.

Hilbert transforms are important in ascertaining the realizability of functions and filters [1], [2], computation of "analytic" signals [3], analysis of single-sideband systems [3], aerodynamics [4], [5], etc. To be able to use digital analysis techniques for the above, a transformation giving a time series of discrete samples for the Hilbert transform would be very welcome, where the input functions are represented as sampled data as can be done for band-limited functions with the help of cardinal series expansion [6]. This transformation has been called the discrete Hilbert transform (DHT).

The DHT of discrete data $f(nT)$, $n=(-\infty,\cdots,-1,0,1,\cdots,\infty)$, is defined by

$$\mathrm{DHT}\{f(nT)\} = g(kT) = \begin{cases} \dfrac{2}{\pi}\displaystyle\sum_{n\ \mathrm{odd}} \dfrac{f(nT)}{k-n}; & k \text{ even} \\ \dfrac{2}{\pi}\displaystyle\sum_{n\ \mathrm{even}} \dfrac{f(nT)}{k-n}; & k \text{ odd.} \end{cases} \tag{1}$$

Thus $g(kT)$ is the kth sample of the transformed time series. The inverse relationship is given by

$$f(nT) = \begin{cases} -\dfrac{2}{\pi}\displaystyle\sum_{k\ \mathrm{odd}} \dfrac{g(kT)}{n-k}; & n \text{ even} \\ -\dfrac{2}{\pi}\displaystyle\sum_{k\ \mathrm{even}} \dfrac{g(kT)}{n-k}; & n \text{ odd.} \end{cases} \tag{2}$$

To show that the relationship (2) is valid, we insert (1) in (2). For n even,

$$f(nT) = -\frac{2}{\pi}\sum_{k\ \mathrm{odd}} \frac{1}{n-k}\,\frac{2}{\pi}\sum_{p\ \mathrm{even}} \frac{f(pT)}{k-p}. \tag{3}$$

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Interchanging the order of summation, a step validated by Fubini's theorem [7], we have

$$f(nT) = +\frac{4}{\pi^2}\sum_{p\ \mathrm{even}}\sum_{k\ \mathrm{odd}} \frac{f(pT)}{(k-n)(k-p)} \tag{4}$$

$$= \frac{4}{\pi^2}\sum_{k\ \mathrm{odd}} \frac{f(nT)}{(k-n)^2} + \frac{4}{\pi^2}\sum_{\substack{p\ \mathrm{even}\\ p\neq n}}\sum_{k\ \mathrm{odd}} f(pT)(n-p) \left\{\frac{1}{k-n} - \frac{1}{k-p}\right\}. \tag{5}$$

The term within the braces is clearly zero when k runs through all the values. The contribution from the first sum is

$$f(nT) = \frac{8}{\pi^2} f(nT)\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right\}. \tag{6}$$

But from [8],

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}. \tag{7}$$

Therefore, the correctness of relation (2) has been established for even n, and similarly for odd n.

To establish the analogy of the DHT with the Hilbert transform for continuous functions, we consider a signal $s(t)$, band-limited to ω_0 rad/s. Its cardinal series expansion would be

$$s(t) = \sum_{n=-\infty}^{+\infty} s(nT)\frac{\sin \omega_0(t-nT)}{\omega_0(t-nT)}, \qquad T = \pi/\omega_0. \tag{8}$$

Its Hilbert transform is

$$\mathscr{H}\{s(t)\} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{s(y)}{t-y}\,dy \tag{9}$$

where the above integral is defined as its Cauchy principal value. Thus we write

$$\mathscr{H}\{s(t)\} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} \frac{1}{(t-y)} s(nT)\frac{\sin \omega_0(y-nT)}{\omega_0(y-nT)}\,dy. \tag{10}$$

This integration can be carried out easily. Otherwise, from tables [9],

$$\mathscr{H}\{s(t)\} = \sum_{n=-\infty}^{+\infty} s(nT)\left\{\frac{1-\cos \omega_0(t-nT)}{\omega_0(t-nT)}\right\}. \tag{11}$$

Since $\mathscr{H}\{s(t)\}$ is also a band-limited function [3], a cardinal series expansion for it can also be written. At the Nyquist periods $t=kT$, we have

$$\mathscr{H}\{s(kT)\} = \sum_{n=-\infty}^{+\infty} s(nT)\left\{\frac{1-\cos \pi(k-n)}{\pi(k-n)}\right\} = \begin{cases} \dfrac{2}{\pi}\displaystyle\sum_{n\ \mathrm{odd}} \dfrac{s(nT)}{k-n}; & k \text{ even} \\ \dfrac{2}{\pi}\displaystyle\sum_{n\ \mathrm{even}} \dfrac{s(nT)}{k-n}; & k \text{ odd.} \end{cases} \tag{12}$$

Thus we note that the DHT of the Nyquist samples gives a new set of samples which may be used to construct the Hilbert transform of the signal directly. We conclude that the Hilbert transformation and the DHT are completely analogous and the theorems defined for the former can be easily extended to include the latter.

If the data set is finite, we assume the signal to be zero at other Nyquist points and calculate the DHT as before.

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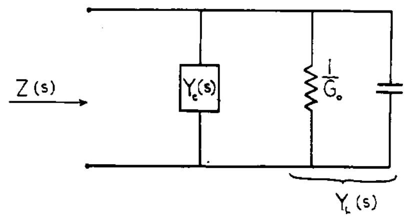

Fig. 1.

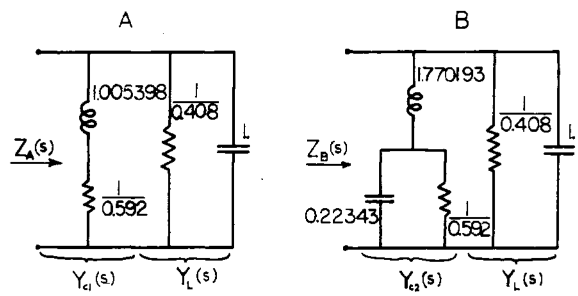

Fig. 2.

A Note on Carlin's Technique for Gain-Bandwidth "Optimization" in Passive One-Ports

This letter refers to Carlin's paper[1] (Section II) on the design of one-port equalizers with a parasitic parallel *RC* load.

Carlin's procedure does not yield the maximum gain-bandwidth product. To see this, one need only invoke the basic Bode's theorem[2] relating gain and bandwidth to conclude that the real part of the added portion of the equalizer network $Y_c(s)$ should not degenerate into a shunt conductance at $\omega \to \infty$ (see Fig. 1) if the gain-bandwidth product is to be maximized. Carlin's compensation violates this condition.

Networks of the superior performance (compared to Carlin's design), realizable under the condition that $\lim_{\omega \to \infty} \mathrm{Re}\,[Yc(j\omega)] = 0$, are shown in Fig. 2. Both networks in Fig. 2 are terminated by the identical parallel *RC* loads used in Carlin's numerical example. The values of the elements in Fig. 2 are normalized with respect to $\omega_{01} = 0.37707 \times 10^6$ rad/s and $C_0 = 6.5 \times 10^{-9}$ farads. Evidently, the dc impedance of both networks in Fig. 2 is 408 Ω after rescaling, i.e., it has the same level as the one in Carlin's example. Hence one need only examine the respective bandwidths in order to compare the performance of the networks in Fig. 2 with Carlin's realization. These results are tabulated in Table I and complete frequency magnitude responses are shown in Fig. 3.

Note that the compensating networks A and B in Fig. 2 contain only two and three elements, respectively, but contribute to a significantly better performance than Carlin's compensation circuitry[1] in Fig. 5 which contains five elements plus a transformer with the unrealistic value of $k = 1$. It is also important to note that the simple compensating network with only three elements in Fig. 2 realizes an even larger -3 dB bandwidth than Carlin's idealized response requiring infinite complexity.

Both networks A and B of Fig. 2 are synthesized on a rigorous and systematic basis and represent unique realizations which are optimum in a gain-bandwidth (-3 dB bandwidth) sense for the prescribed complexity of the respective input impedance functions. The details of the general design procedure will be presented in a future publication.

Finally, besides some obvious typographical errors, it should be pointed out that the substitution for Re $Y_L(j\omega)$ in (12), Section II in the Carlin paper,[1] is not the real part of the *parallel RC* load admittance Carlin has otherwise been concerned with.

TABLE I

	-3 dB Bandwidth	Increase in the Bandwidth of A and B versus C and D
A	1.500980 ω_{01}	7 percent over C
B	1.628058 ω_{01}	16 percent over C, 1.6 percent over D
C	1.40 ω_{01}	
D	1.60 ω_{01}	

A = network A in Fig. 2
B = network B in Fig. 2.
C = Carlin's realization[1] (Figs. 5 and 6)
D = Carlin's ideal (calculated only) response ([1], Fig. 6)
$\omega_{01} = 0.37707 \times 10^6$ rad/s.
Note: Carlin's example[1] (Fig. 6) is normalized with respect to $\omega_0 = 0.5 \times 10^6$ rad/s.

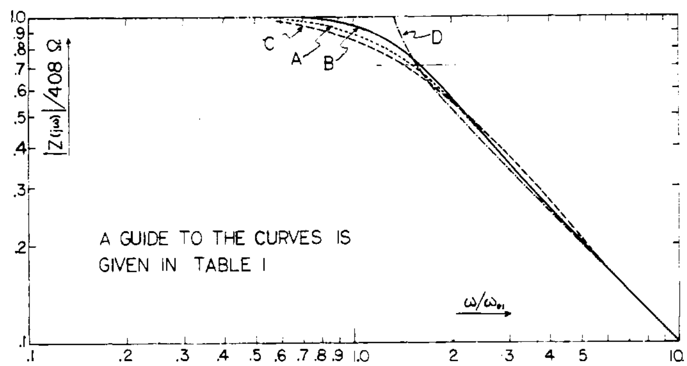

Fig. 3.

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