Transmission Lines

1 Single transmission-line equations

Consider the basic telegrapher's equations for a single ideal (lossless, dispersionless) transmission line as a function of the time t and the position x along the line (see Figure 1).

$$-\frac{\partial V}{\partial x} = L\frac{\partial I}{\partial t} \tag{1a}$$

$$-\frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t} \tag{1b}$$

where V and I are the voltage and current, respectively, and L and C are the inductance and capacitance per unit length. Solutions must be sought for the voltage and current which

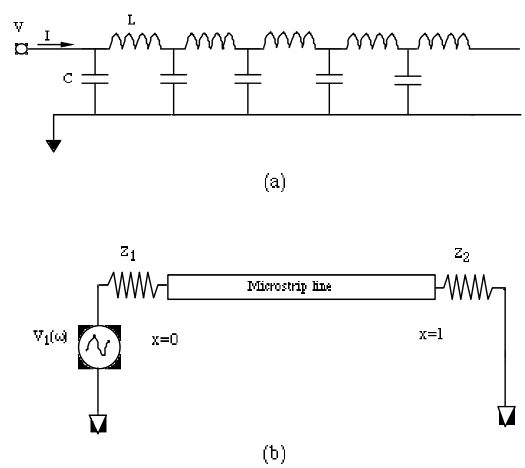

Figure 1. (a) Basic electrical model for lossless transmission line. (b) Representation of a single ideal lossless microstrip line with linear resistive terminations.

satisfy the boundary conditions at x=0 and at x=l, where l is the length of the line. First, Equations (1) are combined to yield the second-order differential equations

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \tag{2a}$$

$$\frac{\partial^2 I}{\partial x^2} = CL\frac{\partial^2 I}{\partial t^2} \tag{2b}$$

For matters of convenience, we need to find a frequency-domain solution for a time-harmonic excitation of frequency $f = \omega/2\pi$ where ω is the angular frequency. This implies that V and I have $e^{j\omega t}$ time dependence; therefore, Equations (2) can be written as

$$\frac{\partial^2 V}{\partial x^2} = -\omega^2 LCV \tag{3a}$$

$$\frac{\partial^2 I}{\partial x^2} = -\omega^2 CLI \tag{3b}$$

The general solutions for the voltage and the current are given by

$$V(x) = Ae^{-j\omega x/v_o} + Be^{+j\omega x/v_o} \tag{4a}$$

$$I(x) = \frac{A}{Z_o}\, e^{-j\omega x/v_o} - \frac{B}{Z_o}\, e^{+j\omega x/v_o} \tag{4b}$$

where the characteristic impedance Z_o is given by

$$Z_o = \sqrt{\frac{L}{C}} \tag{5}$$

and the propagation velocity v_o is given by

$$v_o = \frac{1}{\sqrt{LC}}\ . \tag{6}$$

Equations (4) are superpositions of forward and backward wave solutions to Equations (3) and the coefficients A and B are obtained by matching the boundary conditions at x=0 and x=l (see Figure 1(b)), namely,

$$V_1(\omega) = V(0) + Z_1\, I(0) \tag{7a}$$

$$0 = V(l) - Z_2 I(l)\,. \tag{7b}$$

$V_1(\omega)$ is the source voltage provided at x=0. By substituting the above conditions in Equations (4), we get the values for the coefficients A and B.

$$A = \frac{T\, V_1(\omega)}{1 - \Gamma_1 \Gamma_2 e^{-2j\omega l/v_o}} \tag{8a}$$

$$B = \Gamma_2 e^{-2j\omega l/v_o} A \tag{8b}$$

where T_1 is the transmission coefficient; Γ_1 and Γ_2 are the source and load reflection coefficients respectively; they are given by

$$T = \frac{Z_o}{Z_1 + Z_o} \tag{9}$$

$$\Gamma_1 = \frac{Z_1 - Z_o}{Z_1 + Z_o} \tag{10}$$

$$\Gamma_2 = \frac{Z_2 - Z_o}{Z_2 + Z_o}\ . \tag{11}$$

A and B can be substituted into Equations (4) in order to get the complete solutions. It is of interest to observe that the magnitudes of Γ_1 and Γ_2 are less than 1; therefore, the denominator of Equation (8a) can be expanded into a geometric series which yields

$$A = TV_1(\omega)\sum_{k=0}^{\infty} \Gamma_1^k \Gamma_2^k e^{-2j\omega kl/v_o} \tag{12a}$$

$$B = TV_1(\omega)\sum_{k=0}^{\infty}\Gamma_1^k\Gamma_2^{k+1}e^{-2j\omega(k+1)l/v_o} \tag{12b}$$

Therefore, the frequency-domain solutions are described by forward and backward wave solutions given respectively by

$$V_f(x) = \sum_{k=0}^{\infty}\Gamma_1^k\Gamma_2^k e^{-2j\omega kl/v_o}e^{-j\omega x/v_o}TV_1(\omega) \tag{13a}$$

$$V_b(x) = \sum_{k=0}^{\infty}\Gamma_1^k\Gamma_2^{k+1}e^{-2j\omega(k+1)l/v_o}e^{+j\omega x/v_o}TV_1(\omega) \tag{13b}$$

and the total voltage is given by

$$V(x) = V_f(x) + V_b(x)\,. \tag{14}$$

Several important features need to be stressed regarding Equations (13). First, we notice that for moderate reflection coefficients, only a few terms of the infinite series need to be retained in order to obtain an accurate approximation. Second, by recognizing that $V_1(\omega)$ can be regarded as the Fourier transform of some arbitrary time-domain voltage, $V_1(t)$, and by recalling that $e^{-j\omega\tau}$ is the Fourier transform of the delayed impulse function $\delta(t-\tau)$, inversion of Equation (13) can easily be performed into the time domain:

$$V_f(x,t) = T\sum_{k=0}^{\infty}\Gamma_1^k\Gamma_2^k V_1\left(t - \frac{x+2kl}{v_o}\right) \tag{15a}$$

$$V_f(x,t) = T\sum_{k=0}^{\infty}\Gamma_1^k\Gamma_2^{k+1} V_1\left(t - \frac{x-2(k+1)l}{v_o}\right) \tag{15b}$$

$$V(x,t) = V_f(x,t) + V_b(x,t)\,. \tag{16}$$

Equations (15) and (16) represent the general time-domain solution for an arbitrary excitation, $V_1(t)$. As can be seen, they are made of a superposition of delayed and attenuated components of the original signal $V_1(t)$.