Uniformly Accurate Electrostatic Layered Medium Green's Function Approximation via Scattered Field Formulation

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***Abstract*—A novel approach to evaluation of electrostatic multilayered media Green's functions is presented. Total field of the point charge in layered substrate is represented as a sum of known closed-form incident field in homogeneous space and scattered field which accounts for the effect of the layers. The former accurately approximates the field near the source while the latter is approximated with cylindrical waves and accurately represents the intermediate and far fields. The cylindrical waves approximation is performed via numerical solution of the differential equation formulated with respect to the scattered field as opposed to the total field as it was done in previous work. The spectral domain scattered field solution is cast into the pole-residue form. It allows for the subsequent analytical evaluation of the Sommerfeld integrals producing closed-form space domain approximation.**

***Index Terms*—Green's function, FEM, scattered field formulation**

I. Introduction

Evaluation of the multilayered medium Green's function is vital in many practical applications of electromagnetic analysis including design of microwave circuits and micro strip antennas, modeling of high-speed interconnects. The common practice to obtain the spatial domain Green's function is to solve the 1D spectral domain boundary value problem (BVP) analytically [1], then perform the inverse Fourier-Bessel transform through approximating the Green's function spectrum with known functions allowing for subsequent analytical evaluation of the inverse Fourier-Bessel transform [2], [3]. Depending on the choice of fitting functions, the resultant space domain Green's function loses its accuracy in either near zone [2] or far zone [3]. In the former, the error in the near field is due to spherical waves dominating the solution and being approximated with a counted number of cylindrical waves. In order to develop uniformly accurate approximation, we decompose the spectrum of the total field into the incident and scattered field contributions. The 1D differential equation governing the spectrum of the layered medium Green's function is formulated with respect to the scattered field rather than the total field as it was done previously [2], [4]. The boundary conditions at layer interfaces, based on the continuity of Green's function and the normal component of electric flux density are enforced. As a result, the pole-residue approximation of the scattered field spectrum is obtained which leads to accurate cylindrical wave approximation in the space domain in both the intermediate and far zones, since the singularity of Green's function resides in the incident field. The incident field is subsequently added in the analytical form. Hence, the total field accurately describes the field near the source. The proposed method is similar to [5] in that it approximates the scattered field Green's function with a rational function. However, such approximation is achieved through error-controllable solution of the 1D BVP for the scattered field as opposed to the fitting of its analytically determined spectrum with rational function via VECTFIT procedure [6], latter not being an error-controllable process.

This work was supported by Collaborative Research and Development Grant from NSERC and Manitoba Hydro International.

978-1-7281-6161-7/20/$31.00 © 2020 IEEE

II. Numerical Spectral Domain Scattered Potential Evaluation

Consider a parallel plate waveguide bounded by PEC planes situated at elevations $z = 0$ and $z = d$ along the direction of stratification z. The waveguide is filled with planar layered medium. In each layer, the dielectric is assumed to be homogeneous. For a point charge located on the z axis at elevation z' in layer with permittivity ϵ_{src}, the boundary value problem Green's function (i.e. electrostatic potential due to point charge) is governed by the Poisson's equation in cylindrical coordinates

$$\nabla^2 G^{tot}(\rho, z; z') = -\frac{1}{\epsilon_{src}} \frac{\delta(\rho)}{2\pi\rho} \delta(z - z'). \tag{1}$$

Applying forward Fourier-Bessel transform [1]

$$\tilde{G}^{tot}(\lambda, z; z') = \int_0^\infty G^{tot}(\rho, z; z') J_0(\lambda\rho)\rho d\rho, \tag{2}$$

to both sides of (1), we reduce it to the following ordinary differential equation (ODE) with respect to the Green's function spectrum $\tilde{G}^{tot}$ [2]

$$\frac{d^2}{dz^2}\tilde{G}^{tot}(\lambda,z;z')-\lambda^2\tilde{G}^{tot}(\lambda,z;z')=-\frac{1}{2\pi\epsilon_{src}}\delta(z-z'). \tag{3}$$

The spectral domain Green's function $\tilde{G}^{tot}$ is constrained by the boundary conditions at dielectric interfaces and the bounding PEC plates. The boundary conditions at each ith dielectric interface between the layers with elevation $z_{int,i}$

$$\tilde{G}^{tot}\Big|_{z^+_{int,i}}=\tilde{G}^{tot}\Big|_{z^-_{int,i}},\quad \epsilon_i^+\frac{d\tilde{G}^{tot}}{dz}\Big|_{z^+_{int,i}}=\epsilon_i^-\frac{d\tilde{G}^{tot}}{dz}\Big|_{z^-_{int,i}}, \tag{4}$$

represent the continuity of the electrostatic potential and normal component of the total electric flux, respectively. The PEC boundary conditions for spectral domain Green's function $\tilde{G}^{tot}$ are

$$\tilde{G}^{tot}(\lambda,0;z')=0,\qquad \tilde{G}^{tot}(\lambda,d;z')=0. \tag{5}$$

Both sets of boundary conditions (4) and (5) are resulted from the Fourier-Bessel transform of the corresponding boundary conditions for the Green's function in the spatial domain.

A. Spectral 1D BVP for the Incident Field Green's Function

We define the incident field Green's function as the response to the point charge source in the parallel plate waveguide filled with homogeneous dielectric whose permittivity is ϵ_{src} as in the layered problem. The 1D BVP for the incident field consists of the ODE

$$\frac{d^2}{dz^2}\tilde{G}^{inc}(\lambda,z;z')-\lambda^2\tilde{G}^{inc}(\lambda,z;z')=-\frac{1}{2\pi\epsilon_{src}}\delta(z-z'), \tag{6}$$

in conjunction with the boundary conditions at the PEC planes

$$\tilde{G}^{inc}(\lambda,0;z')=0,\qquad \tilde{G}^{inc}(\lambda,d;z')=0, \tag{7}$$

and boundary conditions at the ith dielectric interface $z=z_{int,i}$, i.e., both the incident potential and the incident electric flux are continuous:

$$\tilde{G}^{inc}\Big|_{z^+_{int,i}}=\tilde{G}^{inc}\Big|_{z^-_{int,i}},\quad \frac{d\tilde{G}^{inc}}{dz}\Big|_{z^+_{int,i}}=\frac{d\tilde{G}^{inc}}{dz}\Big|_{z^-_{int,i}}. \tag{8}$$

B. Spectral 1D BVP for the Scattered Green's Function

The scattered field Green's function is defined as $\tilde{G}^{sca}=\tilde{G}^{tot}-\tilde{G}^{inc}$. Subtracting the left hand side of equation (6) from the left hand side of (3) and performing the same subtraction for their right hand sides produces the following 1D ODE for the scattered field Green's function

$$\frac{d^2}{dz^2}\tilde{G}^{sca}(\lambda,z;z')-\lambda^2\tilde{G}^{sca}(\lambda,z;z')=0. \tag{9}$$

Executing similar subtractions for the left and right hand sides of the boundary condition equations for the total and incident field Green's functions we obtain the boundary conditions for the scattered field

$$\begin{aligned}\tilde{G}^{sca}\Big|_{z^+_{int,i}}&=\tilde{G}^{sca}\Big|_{z^-_{int,i}}\\ \epsilon_i^+\frac{d(\tilde{G}^{sca}+\tilde{G}^{inc})}{dz}\Big|_{z^+_{int,i}}&=\epsilon_i^-\frac{d(\tilde{G}^{sca}+\tilde{G}^{inc})}{dz}\Big|_{z^-_{int,i}}\end{aligned} \tag{10}$$

$$\tilde{G}^{sca}(\lambda,0;z')=0,\qquad \tilde{G}^{sca}(\lambda,d;z')=0. \tag{11}$$

C. Numerical Solution of 1D BVP for Scattered Field Spectrum

A numerical solution of the 1D BVP for the scattered Green's function with Finite Difference (FD) [4] or Finite Element Method (FEM) [7] yields the system of linear equations with respect to the discretized scattered field Green's function $\tilde{\mathbf{G}}^{sca}$

$$([A]+\lambda^2[B])\tilde{\mathbf{G}}^{sca}=\mathbf{b}, \tag{12}$$

where the excitation vector $\mathbf{b}$ has non-zero values at the entries associated with the dielectric interfaces as opposed to the excitation vector in analogous system of algebraic equations occurring in the total field formulation [2] where non-zero values in the excitation vector occur at entries corresponding to the location of the source. Performing eigenvalue decomposition on $[B]^{-1}[A]$ similarly as in [2], we get

$$\tilde{\mathbf{G}}^{sca}=([A]+\lambda^2[B])^{-1}\mathbf{b}=[E]([D]+\lambda^2[U])^{-1}[T]\mathbf{b}, \tag{13}$$

where $[E][D][E]^{-1}=[B]^{-1}[A]$, $[T]=([B][E])^{-1}$. Denoting the set of all indices of nodes at the dielectric interfaces as $\{\mathbb{S}_{int}\}$, from (13) we get ith unknown

$$\tilde{G}_i^{sca}=\tilde{G}^{sca}(\lambda,z_i;z')=\sum_{j\in\mathbb{S}_{int}}b_j\sum_{k=1}^{Np}\frac{E_{ik}T_{kj}}{D_k+\lambda^2}. \tag{14}$$

Expression (14) has the same form regardless the numerical method used to solve the 1D BVP for $\tilde{G}^{sca}$. In case second order FEM is used for the solution, for example,

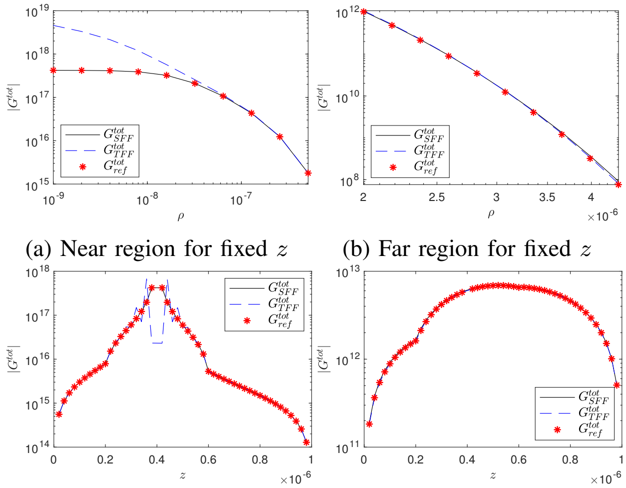
$$b_j=-\frac{2(\epsilon_{i_j+1}-\epsilon_{i_j})}{\epsilon_{i_j+1}+\epsilon_{i_j}}\frac{\tilde{G}^{inc}}{dz}\Big|_{z_{int,i_j}}, \tag{15}$$

where i_j is the index of the medium layer associated with the index j that is the index of the node overlapping the dielectric interface. Performing inverse Fourier-Bessel transform [2] on (14), we have

$$\begin{aligned}G^{sca}(\rho,z_i;z')=\sum_{j\in\mathbb{S}_{int}}C_{i_j}\Bigg(&\sum_{k=1}^{N_p}E_{ik}T_{kj}H_0^{(2)}(-i\sqrt{D_k}\rho)\\ &\sum_{n=1}^{\infty}\frac{g(i_j,n;z')}{D_k-a_n^2}-\sum_{k=1}^{N_p}\sum_{n=1}^{N_t}\frac{g(i_j,n;z')H_0^{(2)}(-ia_n\rho)E_{ik}T_{kj}}{D_k-a_n^2}\Bigg)\end{aligned} \tag{16}$$

where

$$C_{i_j}=-\frac{i(\epsilon_{i_j+1}-\epsilon_{i_j})\pi}{(\epsilon_{i_j+1}+\epsilon_{i_j})\epsilon_{src}d^2},\quad a_n=\frac{n\pi}{d}, \tag{17}$$



(a) Near region for fixed z (b) Far region for fixed z

(c) Near region for fixed ρ (d) Far region for fixed ρ

Fig. 1: Spatial domain total field Green's function behavior in the near and far zones of the point source.

the square root $\sqrt{D_k}$ is chosen such that the field satisfies the radiation condition, and

$$g(i_j, n; z') = n \cos\left(\frac{n\pi z_{int,i_j}}{d}\right) \sin\left(\frac{n\pi z'}{d}\right). \tag{18}$$

In (16), the following series can be analytically evaluated in closed form by Poisson summation formula. For example, for the case when D_k has positive real value and $z_{int,i_j} > z'$,

$$\sum_{n=1}^{\infty} \frac{g(i_j, n; z')}{D_k - a_n^2} =$$
$$\frac{d^2}{8\pi}\left(\frac{e^{-i\sqrt{D_k}(z_{int,i_j}-z')} + e^{i\sqrt{D_k}(z_{int,i_j}+z'-2d)}}{1 - e^{-i2\sqrt{D_k}d}}\right.$$
$$- \frac{e^{-i\sqrt{D_k}(z_{int,i_j}+z')} + e^{i\sqrt{D_k}(z_{int,i_j}-z'-2d)}}{1 - e^{-i2\sqrt{D_k}d}}$$
$$+ \frac{e^{i\sqrt{D_k}(z_{int,i_j}-z')} + e^{-i\sqrt{D_k}(z_{int,i_j}+z'-2d)}}{1 - e^{i2\sqrt{D_k}d}}$$
$$\left.- \frac{e^{i\sqrt{D_k}(z_{int,i_j}+z')} + e^{-i\sqrt{D_k}(z_{int,i_j}-z'-2d)}}{1 - e^{i2\sqrt{D_k}d}}\right).$$

Other cases can be treated analogously and are not shown here due to shortage of space.

The second sum over n in (16) in the bracket was an infinite sum. It can be truncated to a given precision since

$$H_0^{(2)}(-ia_n\rho) = H_0^{(2)}(-i\frac{n\pi}{d}\rho) \sim \sqrt{\frac{2i}{\pi\frac{n\pi}{d}\rho}}\, e^{-a_n\rho} e^{i\frac{\pi}{4}}, \tag{19}$$

is exponentially decaying in magnitude for large number of n. For given error tolerance δ, we keep the first N_t terms, where

$$N_t = \left\lceil -\frac{d \ln \delta}{\rho\pi} \right\rceil. \tag{20}$$

III. Numerical Results

To validate the proposed scattered field formulation, a three layer substrate with total thickness d =1 µm is considered. From bottom to top, the three layers have thicknesses 0.2 µm, 0.4 µm, and 0.4 µm, and relative permittivities $\epsilon_r = 5, 1, 10$, respectively. The source is located at elevation z' =0.6 µm. The behavior of spatial domain total Green's function G^{tot} obtained using the proposed scattered field formulation (SFF) and the total field formulation (TFF) [2] are shown in Fig. 1. In Fig. 1a and 1b, the observation location is fixed as $z = 0.38$ µm, whereas in Fig. 1c, ρ is fixed as $0.001d$ and in Fig. 1d, ρ is fixed at $1.6d$.

One can observe that the proposed scattered field formulation maintains high accuracy of the Green's function approximation both in the vicinity of the source, as can be seen from Figs. 1a and 1c, and far from it, as shown in Figs. 1b and 1d. Note that in Figs. 1c and 1d z spans full thickness, i.e. $z \in [0, d]$. In the far region, both methods provide accurate results manifested in Fig. 1b, where $\rho \in [2d, 4.34d]$, and Fig. 1d, in which ρ is fixed at $1.6d$ and observation z covers entire interval between the PEC plates of the waveguide.

IV. Conclusion

This paper introduces a new methodology for the closed form evaluation of the electrostatic Green's function in shielded planar layered medium. The proposed scattered field formulation of the Green's function allows to construct uniformly accurate Green's function approximation both in the near and far zone of the point source. The method numerically solves the 1D boundary value problem for the spectrum of the scattered field Green's function. It is followed by the eigenvalue decomposition of the pertinent matrices, which allows to cast the spectrum of the scattered field Green's function into the pole-residue form and enable analytic evaluation of the Sommerfeld integrals. The previous version of the method which performed analogous operations on the total field of the Green's function failed to provide accurate field approximation in the near vicinity of the point source.

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