

A Novel Algorithm for Computing the Zeros of Transfer Functions by Local Minima

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Abstract—We present a novel technique for computing the zeros of rational transfer functions in partial fraction form, by finding the local minima of the magnitude of the determinant of a block matrix comprised of state-space sub-matrices and the Laplace variable s . In this paper, the technique is developed for systems with real poles and residues, and successfully applied to a 10^{th} order numerical example. In a separate paper, we further develop the technique to systems with complex-conjugate pairs of poles and residues.

Keywords— *rational function; state space model; local minima; stability*

I. INTRODUCTION

Modern microelectronics design requires extensive signal and power integrity simulations of the interconnect systems, in order to ensure optimal performance. Such analysis [1] of electrical interconnect systems often relies on macromodels which are derived from tabulated frequency response of the system (e.g., scattering, admittance, or impedance parameters) usually derived from electromagnetic simulation or from direct measurement.

Macromodels can be obtained through rational curve fitting of the frequency response [2], which identifies the dominant pole-residue pairs in a partial fraction form. It is often convenient to factor the partial fraction from into a ratio of two polynomials (i.e., numerator over denominator) that expresses the transfer function in terms of poles and zeros, because information about zeros provides quantitative insight into the response characteristics; such as, stability analysis of the system's inverse [3] (e.g., admittance vs. impedance).

Although there are existing methods [4]–[6] for computation of zeros, they require relatively expensive and complex computational operations (e.g., large matrix formations, and singular value decomposition). In this paper, we propose a relatively simple, yet powerful, algorithm based on a block matrix formulation of the transfer function's state-space form, in which the magnitude of the determinant of the block matrix over a finite range of Laplace variable s is computed and searched for the local minima to obtain the location of the zeros. Although the algorithm developed in this paper is based on the partial fraction form of rational transfer functions with real poles and residues, the concept may be extended to complex-conjugate pairs of poles and residues; this is presented in a separate paper. A higher resolution scan may

be performed in the vicinity of the local minima points to get a higher precision zero.

The remainder of this paper is organized as follows. In section II we formulate the block matrix and provide the algorithm for computing zeros. We apply the algorithm to a numerical example in section III, and conclude in section IV.

II. FORMULATION

The most general state-space representation of a linear time-invariant system [7] is,

$$\begin{aligned} \dot{x}(t) &= A.x(t) + B.u(t) \\ y(t) &= C.x(t) + D.u(t) \end{aligned} \quad (1)$$

where the system state is $x(t)$, $u(t)$ is the input signal, $y(t)$ is the output signal and A, B, C, D are system state matrices. This system can be represented by its transfer function matrix [7]

$$H(s) = C.(sI - A)^{-1}B + D \quad (2)$$

Also, the transfer function is a rational function [8] in the complex variable $s = \sigma + j\omega$, that is

$$H(s) = \frac{Q(s)}{P(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad (3)$$

where σ is the attenuation term in units of (rad/s), ω is the angular frequency in units of (rad/s), and imaginary number $j = \sqrt{-1}$.

Zeros are defined as the roots of the polynomial of the numerator of a transfer function and poles are defined as the roots of the denominator of a transfer function. For the generalized transfer function [8],

$$H(s) = \frac{Q(s)}{P(s)} = \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)} \quad (4)$$

where the zeros are z_1, z_2, \dots, z_m and poles are p_1, p_2, \dots, p_n .

For real poles and residues, a system transfer function [9] can be characterized by the partial fraction form as follows,

$$H(s) = \sum_{n=1}^N \frac{c_n}{s - p_n}, \quad (5)$$

Algorithm 1 Compute zeros of $H(s)$.

1. Begin with the system transfer function $H(s)$ in the partial fraction form (5).
 2. Form the block matrix $\Gamma(s)$ as in (6) using poles and residues of the transfer function.
 3. Compute the determinant of the block matrix for different values of σ ranging from 10% below the value of lowest pole to 10% above the highest pole.
 4. Find the local minima locations using condition (7); these are the zeros.
 5. Increase resolution (samples/decade) in the vicinity of each local minima to obtain the desired precision for the zeros.
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where c_n is the n^{th} real residue, p_n is the n^{th} real pole, and N is the total number of poles.

The polynomial $Q(s)$ may be obtained by computing the determinant (Det) of the block matrix [10] $\Gamma(s)$ (6).

$$\Gamma(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \quad (6)$$

where $(sI - A) = \begin{bmatrix} 1-p_1 & 0 & 0 & 0 \\ 0 & 1-p_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & 1-p_n \end{bmatrix} : N \times N$, $B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} : N \times 1$, $C = [1 \ 1 \ \dots \ 1] : 1 \times N$, $D = 0$ and I is the identity matrix of size $N \times N$

The determinant of the $\Gamma(s)$ matrix is the characteristic polynomial $Q(s)$ in equation (3) and (4). The zeros of the transfer function $H(s)$ are the roots of the polynomial $Q(s)$.

In our proposed technique, we vary σ and get data sets of $(s, \Gamma(s))$. If the condition in (7) holds, then we get a local minima.

$$\Gamma(\Psi_{i-1}) - \Gamma(\Psi_i) > 0 \wedge \Gamma(\Psi_{i+1}) - \Gamma(\Psi_i) > 0 \quad (7)$$

where i is the index number, $\Psi_i = i\Delta\sigma$, and $\Delta\sigma$ is the discretization unit along the σ axis. The local minima are the roots of the magnitude of the determinant of $\Gamma(s)$ which imply the zeros of the system.

III. NUMERICAL RESULTS

In this section we demonstrate the proposed technique. Let us consider the following transfer function with 10 distinct poles described in [5].

$$H(s) = \frac{Q(s)}{P(s)} = \frac{6.29342}{s+541} + \frac{8.97702}{s+7919} + \frac{11.5591}{s+104729} + \frac{14.0777}{s+1.29971 \times 10^6} + \frac{16.5554}{s+1.54859 \times 10^7} + \frac{19.0053}{s+1.79425 \times 10^8} + \frac{21.4353}{s+2.03807 \times 10^9} + \frac{23.8501}{s+2.28018 \times 10^{10}} + \frac{26.2531}{s+2.52098 \times 10^{11}} + \frac{3.3673}{s+29} \quad (8)$$

To get the zeros of the transfer function, we vary σ from -10^{12} to -10 while keeping $\omega = 0$. Now, using Algorithm 1, we get 9 local minima which are shown in Figure 1

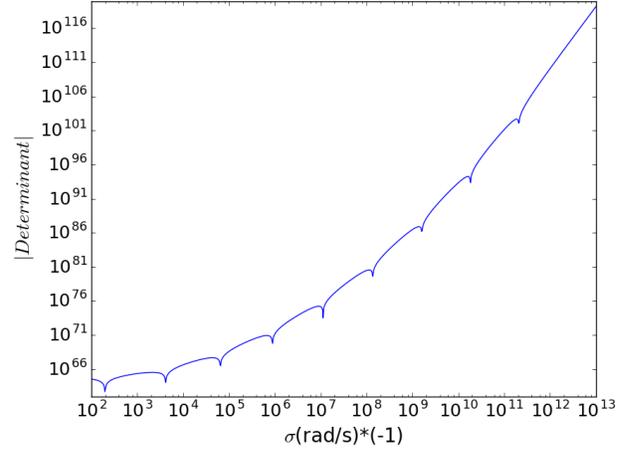


Figure 1. σ vs $|Det(\Gamma(s))|$ Plot

To obtain higher precision values for the location of zeros, we apply step 5 of Algorithm 1 and increase sample numbers per decade. With the higher sampling resolution, we get better approximations of the zeros. The values are given in Table I and II

Table I
VALUE OF ZEROS WITH INCREASED NUMBER OF SAMPLES

10 samples/decade	100 samples/decade
-215.443469	-200.92330026
-4641.58883361	-4229.24287439
-59948.42503189	-65793.32246576
-774263.68268113	-890215.08544504
-11672238.64478847	-11233240.32978027
-1.29154967e+08	-1.38488637e+08
-1.66810054e+09	-1.59228279e+09
-1.675788578e+10	-1.83073828e+10
-2.456987705e+11	-2.10490414e+11

Table II
VALUE OF ZEROS WITH INCREASED NUMBER OF SAMPLES(CONTD.)

1000 samples/decade	10000 samples/decade
-199.66424501	-199.72390234
-4193.94395567	-4189.50020707
-65285.21141128	-65250.14155006
-891148.23228402	-891035.46361709
-11299339.38033222	-11308508.80543817
-1.37131472e+08	-1.36997825e+08
-1.60770442e+09	-1.60923942e+09
-1.84614695e+10	-1.83073828e+10
-2.09083769e+11	-2.10490414e+11

As the number of samples/decade increases, we get closer to the exact zero values given in [5] which are shown in Figure 2, 3 and 4 for $z_2 = -4189.67$

IV. CONCLUSION

We present a simple, yet powerful, technique for computation of the zeros of a rational transfer function in partial fraction form. We develop an algorithm which uses the block matrix comprised of system's state-space sub-matrices and searches for local minima of the magnitude of the determinant of the block matrix to find the zeros. The algorithm is successfully demonstrated for a 10th order numerical example with real poles and residues. In a separate paper, we further develop the technique for systems with complex-conjugate pairs of poles and residues.

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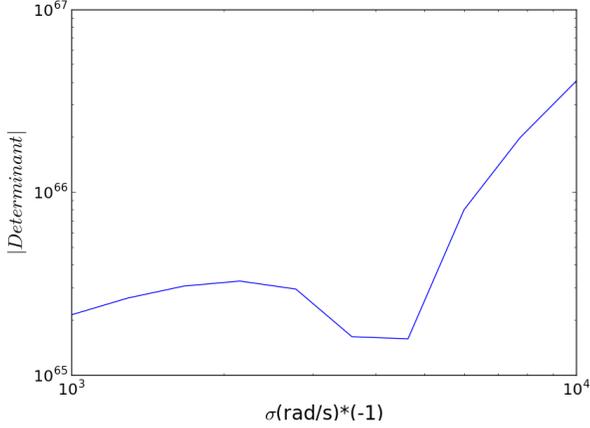


Figure 2. Using 10 samples/decade between $\sigma = -10^4$ to -10^3

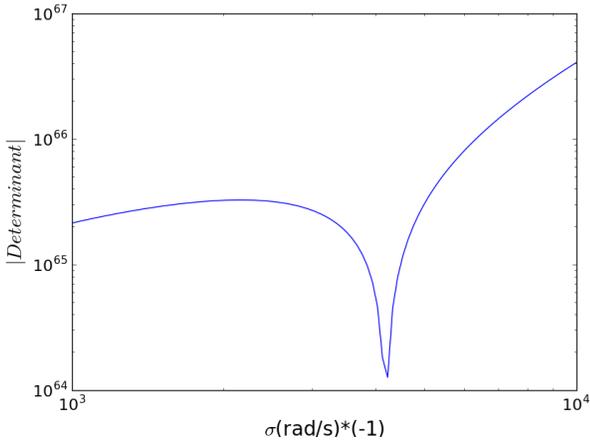


Figure 3. Using 100 samples/decade between $\sigma = -10^4$ to -10^3

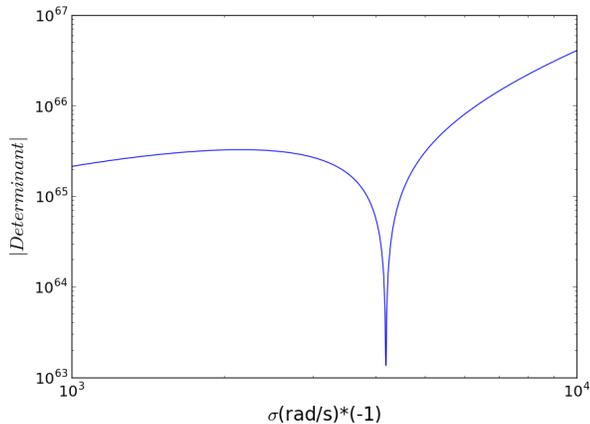


Figure 4. Using 1000 samples/decade between $\sigma = -10^4$ to -10^3