Fast Parameter Extraction for Transmission Lines with Arbitrarily-Shaped Conductors and Dielectrics Using the Contour Integral Method

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***Abstract*—This paper presents an accurate surface formulation based on the contour integral method and the Dirichlet-to-Neumann operator to calculate the impedance and admittance parameters of transmission lines of arbitrary shape. The formulation only requires a discretization of the boundaries of the conductors and dielectrics, as opposed to the entire cross-section, which results in fast computations.**

***Index Terms*—Admittance calculation, surface method, contour integral method.**

I. Introduction

Multiconductor transmission line models are important for the signal integrity analysis of digital interconnects. To create a transmission line model, one first needs to compute the line impedance and admittance over a wide frequency range.

Impedance calculations carried out using standard volumetric methods based on the 2D finite-element method (FEM) or 2D volumetric integral equation methods are slow due to the fine mesh required to model skin depth inside conductors. Surface methods such as those based on the Dirichlet-to-Neumann operator [1] reduce the computational complexity as they require only the discretization of the conductors surface. Previously, the Dirichlet-to-Neumann operator was limited to canonical geometries such as rectangular, triangular, and circular. Recently, we developed a technique [2] based on the contour integral method (CIM) [3] to obtain the surface operator for conductors of arbitrary shape. This new method calculates the per-unit length (p.u.l.) resistance and admittance of a transmission line made by conductors of arbitrary shape using only a contour discretization.

This paper extends the concept of [2] to present a surface formulation based on the Dirichlet-to-Neumann operator to calculate the p.u.l. capacitance of transmission lines. With respect to existing surface methods for capacitance calculation that require volumetric mesh, the Dirichlet-to-Neumann approach only requires discretization of dielectric surface. We generalize the Dirichlet-to-Neumann formulation presented in [4], which can only handle rectangular and triangular geometries, to dielectrics and conductors of arbitrary shapes immersed in a homogeneous or multilayered background medium.

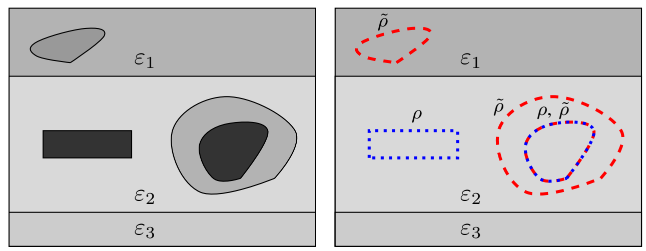

Fig. 1. Left-panel: Sample cross-section with two conductors (black) and two dielectrics (light grey) inside a three-layered dielectric medium. Each layer has different permittivity. Right-panel: Equivalent configuration after all conductors and dielectrics are replaced by equivalent conductor charge density ρ (in blue) and equivalent dielectric charge density $\tilde{\rho}$ (in red).

II. Capacitance Calculation via the CIM

We consider a transmission line made up of P conductors and D dielectrics made up of arbitrary shapes. The background medium can be homogeneous or layered dielectric medium, as shown in the left panel of Fig. 1. Our goal is to calculate the *partial* p.u.l. complex capacitance matrix $\widetilde{\mathbf{C}} = \mathbf{C} + \mathbf{G}/j\omega$ of the transmission line which relates the scalar potential V_c and total charge Q_c on the c-th conductor as

$$\mathbf{Q} = \widetilde{\mathbf{C}}\mathbf{V} \tag{1}$$

where $\mathbf{Q} = \begin{bmatrix} Q_1 & \dots & Q_P \end{bmatrix}^T$ and $\mathbf{V} = \begin{bmatrix} V_1 & \dots & V_P \end{bmatrix}^T$.

A. Discretization of Scalar Potential

We discretize all boundaries of the conductors with N_c segments, and all of the remaining boundaries of the dielectrics with N_d segments. Boundaries of the layered background medium are not discretized. We expand the scalar potential on all boundaries using pulse basis functions

$$V(\mathbf{r}) = \sum_{n=1}^{N_c+N_d} v_n \Pi_n(\mathbf{r}) \,, \tag{2}$$

where $\Pi_n(\mathbf{r})$ is equal to one if $\mathbf{r}$ belongs to the n-th segment, and is zero otherwise. The expansion coefficients in (2) are now collected into vector

$$\mathbf{\Phi} = \begin{bmatrix} v_1 & v_2 & \dots & v_{N_c+N_d} \end{bmatrix}^T \,. \tag{3}$$

B. Equivalent Charge Density

In order to calculate the admittance, we invoke the concept of contrast charge density presented in [4]. We replace all the conductors and dielectrics by the surrounding medium and introduce the so-called equivalent contrast charge densities $\rho(\mathbf{r})$ and $\tilde{\rho}(\mathbf{r})$ on the boundary of all conductors and dielectrics, respectively, as shown in the right panel of Fig. 1. We expand both equivalent charge densities using pulse basis functions as

$$\rho(\mathbf{r}) = \sum_{n=1}^{N_c} \rho_n \Pi_n(\mathbf{r})\,, \tag{4}$$

$$\tilde{\rho}(\mathbf{r}) = \sum_{n=1}^{N_d} \tilde{\rho}_n \Pi_n(\mathbf{r})\,. \tag{5}$$

We collect the coefficients of (4) and (5) into vectors

$$\mathbf{R} = \begin{bmatrix}\rho_1 & \rho_2 & \dots & \rho_{N_c}\end{bmatrix}^T\,, \tag{6}$$

$$\widetilde{\mathbf{R}} = \begin{bmatrix}\tilde{\rho}_1 & \tilde{\rho}_2 & \dots & \tilde{\rho}_{N_d}\end{bmatrix}^T\,. \tag{7}$$

1) Conductors: When a charge is placed on a conductor with high conductivity, it moves to conductor's boundary very quickly. So, contrast charge density on good conductors is actually equal to the free charge density on the conductor surface, and is related to the total charge Q_c by

$$Q_c = \oint_{\gamma_c} \rho(\mathbf{r}')dc'\,, \tag{8}$$

where the integration is performed over the closed contour γ_c enclosing the c-th conductor. The discrete counterpart of (8) for all conductors may be written as

$$\mathbf{Q} = \mathbf{W}\mathbf{R}\,. \tag{9}$$

2) Dielectrics: The contrast charge density on the boundary γ_d of the d-th dielectric is related to the potential $V(\mathbf{r})$ by [4]

$$\tilde{\rho}(\mathbf{r}) = \left(\varepsilon - \varepsilon_b + \frac{\sigma}{j\omega}\right)\frac{\partial V(\mathbf{r})}{\partial n}\,, \tag{10}$$

where ε and σ are the permittivity and conductivity of the dielectric, and ε_b is the permittivity of the background medium. In [4], eigenfunction expansion is used to relate $V(\mathbf{r})$ and $\partial V(\mathbf{r})/\partial n$. Instead, here we use the contour integral method to relate the scalar potential and its normal derivative. Inside the dielectric medium, scalar potential $V(\mathbf{r})$ satisfies

$$\nabla^2 V(\mathbf{r}) = 0\,. \tag{11}$$

Following the application of Green's identity [5], we can show that the solution of (11) satisfies [3]

$$\oint_{\gamma_d}\left[V(\mathbf{r}')\frac{\partial \ln(\mathbf{r},\mathbf{r}')}{\partial n'} - \ln(\mathbf{r},\mathbf{r}')\frac{\partial V(\mathbf{r})}{\partial n'}\right]dr' = \pi V(\mathbf{r})\,. \tag{12}$$

We discretize (12) with the method of moments and pulse basis functions to obtain

$$\widehat{\mathbf{\Phi}}_d = \mathbf{H}\mathbf{\Phi}_d \tag{13}$$

where $\mathbf{\Phi}_d$ is a subset of $\mathbf{\Phi}$ containing potentials on the boundary γ_d, and $\widehat{\mathbf{\Phi}}_d$ is a vector that contains the expansion coefficients of the normal derivative of scalar potential on γ_d. Finally, by substituting (13) into discretized form of (10) for all dielectrics in the system, we obtain

$$\widetilde{\mathbf{R}} = \mathbf{Y}\mathbf{\Phi}\,, \tag{14}$$

where $\mathbf{Y}$ is the desired surface operator, which compactly relates contrast charge density to potentials.

C. Capacitance Calculation

The equivalent charge densities and scalar potential $V(\mathbf{r})$ are related by

$$V(\mathbf{r}) = -\frac{1}{\varepsilon}\int_{\sum\gamma_c} \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dc' - \frac{1}{\varepsilon}\int_{\sum\gamma_d}\tilde{\rho}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dc' \tag{15}$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function of the background medium. We discretize (15) using the method of moments obtaining

$$\mathbf{\Phi} = \begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}\begin{bmatrix}\mathbf{R}\\ \widetilde{\mathbf{R}}\end{bmatrix} \tag{16}$$

where $\mathbf{G}$ is the Green's matrix. By substituting (14) into (16), we obtain

$$\mathbf{\Phi} = \left(\mathbf{1} - \mathbf{G}_2\mathbf{Y}\right)^{-1}\mathbf{G}_1\mathbf{R}\,, \tag{17}$$

which relates conductor potentials with equivalent charge density on the conductors. In (17), $\mathbf{1}$ is the identity matrix. At this point, it is straightforward to find $\widetilde{\mathbf{C}}$ by applying a constant potential on the conductors, and solving (17) to find the total charge induced on each conductor.

III. Impedance Calculation via the CIM

In Sec. II-B2, we used the contour integral equation (12) to find the capacitance by relating $\rho(\mathbf{r})$ and $V(\mathbf{r})$. The dual problem is to find the p.u.l. impedance, relating the electric field and current using the contour integral equation. Similar to the capacitance case, we replace all conductors by their surrounding medium and equivalent currents $J_s(\mathbf{r})$ on conductors boundary. The current $J_s(\mathbf{r})$ on the boundary of each conductor is given by [1]

$$J_s(\mathbf{r}) = \frac{1}{j\omega}\left[\frac{1}{\mu}\frac{\partial E_z(\mathbf{r})}{\partial n} - \frac{1}{\mu_l}\frac{\partial \widetilde{E}_z(\mathbf{r})}{\partial n}\right]_{\mathbf{r}\text{ on }\gamma_c} \tag{18}$$

where E_z and $\widetilde{E}_z$ are, respectively, the electric fields on the boundary of the conductor before and after it is replaced by the background medium. Both E_z and $\widetilde{E}_z$ satisfy

$$\nabla^2 E_z(\mathbf{r}) + k^2 E_z(\mathbf{r}) = 0\,, \tag{19}$$

inside the conductor, where $k = \sqrt{\omega\mu\left(\omega\varepsilon - j\sigma\right)}$ is the wavenumber. In order to obtain the surface operator, we invoke the contour integral equation for the Helmholtz equation [3]

$$E_z(\mathbf{r}) = \frac{j}{2}\oint_{\gamma_c}\left[\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}E_z(\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\frac{\partial E_z(\mathbf{r}')}{\partial n'}\right]dr'\,, \tag{20}$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function associated with the Helmholtz equation in 2D medium. By discretizing (20) using

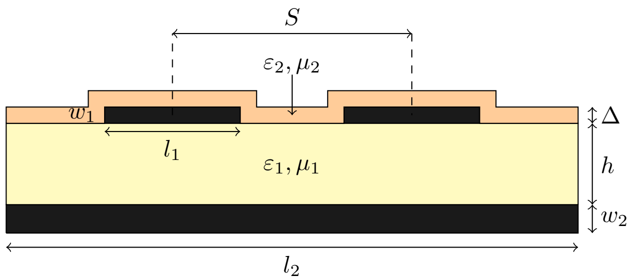

Fig. 2. Coated differential pair example considered in Sec. IV, with $l_1 = 166\ \mu$m, $w_1 = 17.4\ \mu$m, $w_2 = 34.8\ \mu$m, $S = 292.4\ \mu$m, $h = 100.9\ \mu$m.

the method of moments, we obtain a relationship between E_z and $\partial E_z / \partial n$ on the boundary of the conductors. This relationship is then combined with (18) to obtain

$$J_s = \mathcal{Y} E_z \,, \tag{21}$$

where $\mathcal{Y}$ is the surface admittance operator that efficiently captures skin effect inside conductors of arbitrary shape with only contour discretization. Readers may refer to [2] for details on the use of (21) to calculate the impedance parameters.

IV. Numerical Results

We consider the coated differential line in Fig. 2. The dimensions of the system are given in Fig. 2 and were obtained from [6]. Table I shows the capacitance calculated for various values of dielectric permittivities and geometrical parameters with the proposed method and with FEM [7]. The results are in excellent agreement with self and mutual capacitance errors always lower than 1 pF/m. The computational time with the proposed method was 0.80 s, as opposed to FEM which took 8 s.

Figure 3 shows the p.u.l. resistance and inductance parameters calculated for $l_2 = 525\ \mu$m. The plot shows an excellent agreement between the proposed method and FEM [7]. Impedance computation with the proposed method required a total of 266 pulse basis functions, as opposed to the FEM simulation which required 26,170 basis functions. The proposed technique required 0.24 s per frequency point, as opposed FEM [7] which required 9.05 s.

The proposed method led to a speed-up with respect to FEM of 10X and 38X in the extraction of capacitance and resistance/inductance, respectively. These results demonstrate the merit of the proposed idea, and its potential for accelerating parameter extraction in more complex 2D and 3D scenarios that are currently under investigation.

V. Conclusions

We presented a surface method to efficiently extract the capacitance, resistance and inductance of multiconductor transmission lines. The method is based on the contour integral method, which provides an elegant way to generalize the Dirichlet-to-Neumann approach to conductors and dielectrics of arbitrary shape. Since the proposed method requires only a discretization of the surface of conductors and dielectrics, it is more computationally efficient than volumetric techniques like the finite element method.

TABLE I
Example of Sec. IV: Capacitance values (in pF/m) calculated with the proposed technique and with FEM [7].

ε_1	ε_2	Δ	l_2 [μm]	Proposed C_{11}	Proposed C_{12}	FEM C_{11}	FEM C_{12}	Error [pF/m]
2.94	1	-	525	76.1	-8.6	76.0	-9.0	0.4
4.3	1	-	525	104.3	-9.6	104.4	-10.0	0.4
4.3	1	-	1050	112.1	-8.6	112.8	-8.7	0.7
4.3	3.2	w	1050	123.1	-12.3	123.1	-12.6	0.3
4.3	3.3	2w	1050	128.5	-15.0	128.7	-15.4	0.4

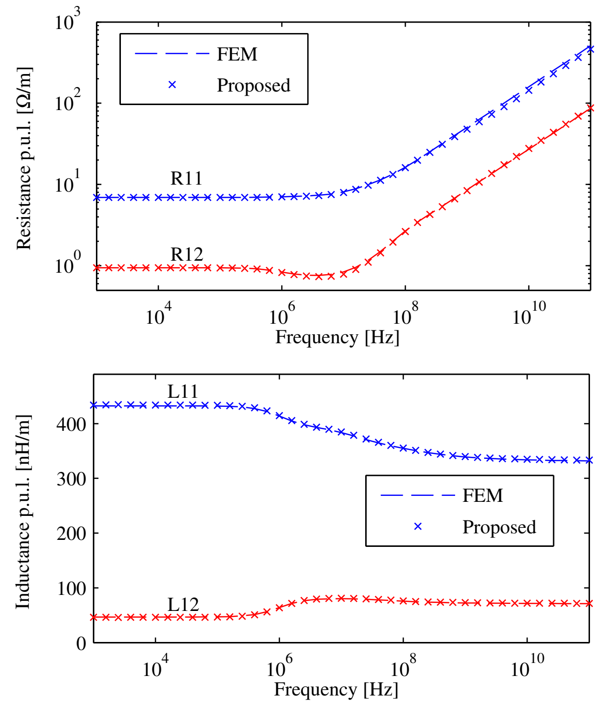

Fig. 3. Per-unit length resistance (top panel) and inductance (bottom panel) obtained with the proposed method and FEM [7] for the example considered in Sec IV

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