A Stochastic Collocation Technique for Time-Domain Variability Analysis of Active Circuits

K. Guo[1], F. Ferranti[2], B. Nouri[1] and M. Nakhla[1]
[1] Carleton University, Ottawa, Canada K1S 5B6
[2] Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium
Email: [1]KaiGuo3@cmail.carleton.ca, [2]francesco.ferranti@vub.ac.be, [1]{sbnouri, msn}@doe.carleton.ca

***Abstract*—A novel method is presented for time-domain statistical analysis of large active circuits with multiple stochastic parameters. It is based on a stability-preserving model order reduction algorithm coupled with stochastic collocation schemes. Pertinent numerical results validate the proposed method.**

I. Introduction

Predicting the effect of the variability of design parameters on the performance of high-speed integrated circuits is of paramount importance for a successful design. Monte Carlo (MC) analysis gives accurate results and its implementation is straightforward, but it requires a large number of simulations and the results are computationally expensive. Alternative approaches have been proposed in the literature.

Variability analysis techniques based on the notion of polynomial chaos (PC) have been investigated over the years [1]–[3]. PC-based modeling approaches expand a circuit response in a series of Askey-Wiener type of orthogonal polynomials and a Galerkin projection step is used to generate an augmented deterministic system of equations whose solution becomes computationally expensive for an increasing number of stochastic parameters and large circuits. Although some methods have been proposed to mitigate this issue [4], [5], the curse of dimensionality remains a major obstacle in using the PC-based techniques.

Another class of techniques based on the stochastic collocation (SC) method has been proposed in the literature [6], [7]. SC schemes approximate the unknown stochastic solution by a polynomial interpolation function in the multidimensional space of the stochastic variables. The interpolation is constructed by repeatedly solving (sampling) the deterministic problem at a predetermined set of nodes. This approach offers high accuracy, fast convergence rate and easy implementation. The computational effort required for a stochastic collocation approach depends on the number of support nodes used for constructing the interpolation. No augmented system of equations is generated.

Recently, an approach has been proposed for time-domain statistical analysis of large circuits with multiple stochastic parameters [8]. This technique is based on parameterized model order reduction (PMOR) and Numerical Inversion of Laplace Transform (NILT) [9]. The algorithm is highly parallelizable in comparison with time-stepping numerical integration techniques and does not require an explicit presentation of the dynamic model in the form of a set of differential equations. However, this technique is only applicable to passive circuits, while there are many practical circuits that are stable but not passive. These circuits generally include dependent voltage and/or dependent current sources e.g. in small-signal device models. Using conventional MOR techniques such as PRIMA [10] for these circuits will not guarantee the stability of the reduced model. Unstable models can lead to inaccurate or totally unfeasible time-domain simulations.

In this paper, we propose a novel method for efficient time-domain variability analysis of large active circuits with multiple stochastic parameters. Three main key features are the core of this method: 1) SC schemes; 2) an algorithm for stability-preserving MOR of active circuits; 3) NILT. The SC algorithm selects the nodes in the space of the stochastic parameters in a parsimonious way by means of sparse grids. Each node corresponds to a set of equations for the active circuit under study. The equations are reduced by the MOR technique [11] and NILT is used to compute the time-domain response of the reduced model at each node. The MOR scheme is used to further speed-up the NILT computation. These time-domain responses are finally used by the stochastic collocation scheme to compute stochastic quantities (e.g., mean and standard deviation).

The proposed variability analysis technique is described in Section II. Numerical results and conclusions are presented in Sections III and IV, respectively.

II. Proposed variability analysis technique

In this section, an overview of the proposed variability analysis algorithm is presented. The basic idea of SC schemes is to approximate the unknown stochastic solution by a polynomial interpolation function in the multidimensional random space. A stochastic solution $\mathbf{y}(t,\xi)$ can be expressed as

$$\mathbf{y}(t,\xi) \approx \mathcal{F}_i(\mathbf{y}) = \sum_{j=1}^{N} \mathbf{y}(t,\xi_j) L_j(\xi) \tag{1}$$

where t represents the time variable, ξ denote a stochastic parameter and $\{L_j(\xi)\}_{j=1}^{N}$ represents the interpolation basis functions. The computational effort required for the SC approach depends on the number of nodes $\xi_j,\ j = 1, ..., N$ used to construct the interpolation. The stochastic collocation procedure reduces to solving a set of N decoupled deterministic systems at each node $\xi_j,\ j = 1, ..., N$. The polynomial interpolation can be expressed by using the Lagrange interpolation polynomials [6], [12]. The nodes can be chosen according to quadrature rules in tensor product or sparse grids. A generalization to multiple stochastic parameters $\boldsymbol{\xi} = (\xi^{(1)}, \ldots, \xi^{(d)})$ can be easily obtained using tensor product grids [12], [13]

$$\mathbf{y}(t, \xi^{(1)}, \cdots, \xi^{(d)}) \approx \mathcal{F}_{\boldsymbol{i}}(\mathbf{y}) = \sum_{j_1=1}^{N_1} \cdots \sum_{j_d=1}^{N_d} \mathbf{y}(t, \xi_{j_1}^{(1)}, \cdots, \xi_{j_d}^{(d)}) L_{j_1}(\xi^{(1)}) \cdots L_{j_d}(\xi^{(d)}) \quad (2)$$

The m-th stochastic moment of $\mathbf{y}(t, \boldsymbol{\xi})$ can be expressed as

$$\mathbb{E}[\mathbf{y}^m(t, \xi^{(1)}, \ldots, \xi^{(d)})] = \int_{\Gamma_1} \cdots \int_{\Gamma_d} \mathbf{y}^m(t, \xi^{(1)}, \ldots, \xi^{(d)}) W(\xi^{(1)}, \ldots, \xi^{(d)}) \approx \sum_{j_1=1}^{N_1} \ldots \sum_{j_d=1}^{N_d} \mathbf{y}^m(t, \xi_{j_1}^{(1)}, \ldots, \xi_{j_d}^{(d)}) w_{j_1}^{(1)}, \ldots, w_{j_d}^{(d)} = \mathcal{Q}_{\boldsymbol{i}}(\mathbf{y}^m) \quad (3)$$

$$w_p^{(r)} = \int_{\Gamma_r} L_p(\xi^{(r)}) W(\xi^{(r)}) d\xi^{(r)} \quad (4)$$

where $\boldsymbol{W}(\boldsymbol{\xi}) = W(\xi^{(1)}) \cdots W(\xi^{(d)})$ is a joint probability density function with support $\Gamma_1 \times \cdots \times \Gamma_d$. By using sparse grids constructed by the Smolyak algorithm, the curse of dimensionality associated with full tensor product collocation grids is reduced. Sparse grids are constructed from a linear combination of tensor product grids with a relatively small numbers of grid points while preserving a high level of accuracy [14]. Using sparse grids, (2) and (3) can be rewritten as [12], [13]:

$$\mathbf{y}(t, \xi^{(1)}, \ldots, \xi^{(d)}) \approx \mathcal{A}_{q,d}(\mathbf{y}) = \sum_{q-d+1 \le |\boldsymbol{i}| \le q} (-1)^{q-|\boldsymbol{i}|} \binom{d-1}{q-|\boldsymbol{i}|} \mathcal{F}_{\boldsymbol{i}}(\mathbf{y}) \quad (5)$$

$$\mathbb{E}[\mathbf{y}^m(t, \xi^{(1)}, \ldots, \xi^{(d)})] \approx \sum_{q-d+1 \le |\boldsymbol{i}| \le q} (-1)^{q-|\boldsymbol{i}|} \binom{d-1}{q-|\boldsymbol{i}|} \mathcal{Q}_{\boldsymbol{i}}(\mathbf{y}^m) \quad (6)$$

where $|\boldsymbol{i}| = i_1 + i_2 + \ldots + i_d$ and q is denoted as the accuracy level of the sparse grid.

The notion of uncertainty in the circuit response due to uncertainty in the parameters is captured by adapting the modified nodal analysis (MNA) formulation [15] at node $\boldsymbol{\xi_j} = (\xi_{j_1}^{(1)}, \ldots, \xi_{j_d}^{(d)})$ as

$$\mathbf{C}(\boldsymbol{\xi_j}) \frac{d}{dt} \mathbf{x}(t, \boldsymbol{\xi_j}) + \mathbf{G}(\boldsymbol{\xi_j}) \mathbf{x}(t, \boldsymbol{\xi_j}) = \mathbf{B} \mathbf{e}(t)$$
$$\mathbf{y}(t, \boldsymbol{\xi_j}) = \mathbf{L} \mathbf{x}(t, \boldsymbol{\xi_j}) \quad (7)$$

where $\mathbf{C}, \mathbf{G} \in \mathbb{R}^{n \times n}$ are parameter-dependent conductance and susceptance matrices, respectively; $\mathbf{x} \in \mathbb{R}^n$ is the vector of MNA variables, $\mathbf{e}(t) \in \mathbb{R}^{n_{in}}$ is the vector of input signals, $\mathbf{y}(t, \boldsymbol{\xi_j}) \in \mathbb{R}^{n_{out}}$ is the parameter-dependent output response, and $\mathbf{B} \in \mathbb{R}^{n \times n_{in}}$ and $\mathbf{L} \in \mathbb{R}^{n_{out} \times n}$ are constant input and output selection matrices, respectively. The active circuit presented by (7) is assumed to be asymptotically stable.

Next, the stability-preserving MOR algorithm [11] is used for constructing a stable reduced macromodel using different left and right projection matrices $\mathbf{U}$ and $\mathbf{V}$ as

$$\widetilde{\mathbf{C}}(\boldsymbol{\xi_j}) \frac{d}{dt} \mathbf{z}(t, \boldsymbol{\xi_j}) + \widetilde{\mathbf{G}}(\boldsymbol{\xi_j}) \mathbf{z}(t, \boldsymbol{\xi_j}) = \widetilde{\mathbf{B}} \mathbf{e}(t)$$
$$\widetilde{\mathbf{y}}(t, \boldsymbol{\xi_j}) = \widetilde{\mathbf{L}} \mathbf{z}(t, \boldsymbol{\xi_j}) \quad (8)$$

where $\mathbf{x} = \mathbf{V}\mathbf{z}$, $\mathbf{z} \in \mathbb{R}^q$, $\widetilde{\mathbf{C}} \triangleq \mathbf{U}^T \mathbf{C} \mathbf{V}$, $\widetilde{\mathbf{G}} \triangleq \mathbf{U}^T \mathbf{G} \mathbf{V} \in \mathbb{R}^{q \times q}$, $\widetilde{\mathbf{B}} \triangleq \mathbf{U}^T \mathbf{B} \in \mathbb{R}^{q \times n_{in}}$, and $\widetilde{\mathbf{L}} \triangleq \mathbf{L}\mathbf{V} \in \mathbb{R}^{n_{out} \times q}$ with $q \ll n$. The right orthogonal projection matrix $\mathbf{V} \in \mathbb{R}^{n \times q}$ is formed through implicitly matching the first q moments in (8) and (7) [10]. The left-projection matrix $\mathbf{U} \in \mathbb{R}^{n \times q}$ is constructed as a function of the right projection matrix through implicitly satisfying a stability condition in the form of a generalized-Lyapunov inequality [11] as $\mathbf{U} = \hat{\mathbf{\Gamma}} \hat{\mathbf{\Gamma}}^{*\mathrm{T}} \mathbf{C} \mathbf{V}$ where $\hat{\mathbf{\Gamma}}$ consists of the q generalized eigenvectors of the matrix pencil $\left(-\mathbf{G}^T, \mathbf{C}^T\right)$. The resulting reduced stable model (8) accurately approximates the input-output behavior and preserves the stability of the original MNA model at each node $\boldsymbol{\xi_j}$ in the grid.

Given a complex frequency-domain response of the reduced model $\widetilde{\mathbf{Y}}(s, \boldsymbol{\xi_j})$, the corresponding time-domain response $\widetilde{\mathbf{y}}(t, \boldsymbol{\xi_j})$ is obtained using the inverse Laplace transformation as

$$\widetilde{\mathbf{y}}(t, \boldsymbol{\xi_j}) = \frac{1}{2\pi j t} \lim_{\omega \to \infty} \int_{(\alpha - j\omega)}^{(\alpha + j\omega)} \widetilde{\mathbf{y}}(\frac{z}{t}, \boldsymbol{\xi_j}) e^z dz \quad (9)$$

where $z \triangleq st$, α is an arbitrary positive constant such that $\Re e\{p_i\} < \alpha$ and p_i denotes the poles of $\widetilde{\mathbf{y}}(s, \boldsymbol{\xi_j})$. Using Padé approximation of the exponential function in (9), the circuit response at each node $\widetilde{\mathbf{y}}(t, \boldsymbol{\xi_j})$ for a given time point t can be evaluated as a linear combination of the responses $\widetilde{\mathbf{y}}(s_i, \boldsymbol{\xi_j})$ at M complex frequencies $s_i \in \{z_1/t, \ldots, z_M/t\}$ as

$$\widetilde{\mathbf{y}}(t, \boldsymbol{\xi_j}) = \frac{-1}{t} \sum_{i=1}^{M} \Re e \left\{ \hat{k}_i \widetilde{\mathbf{y}}(s_i, \boldsymbol{\xi_j}) \right\} \quad (10)$$

where z_i and k_i are the poles and corresponding residues of Padé approximation of order W, $M = W/2$ when W is even; and $M = (W+1)/2$, when M is odd, and $\hat{k}_i = 2k_i$. In this case, $\hat{k}_i = k_i$ for the residue corresponding to the real pole.

Finally, stochastic quantities (e.g., mean and standard deviation) of these time-domain responses are computed based on the previously discussed SC formulas.

III. Numerical Example

In this example, we consider a stable active circuit shown in Fig. 1. The input signal $e(t)$ is chosen as a $2V$ pulse signal with $0.1ns$ fall/rise time and $2.4ns$ pulse-width. Six stochastic physical parameters were considered whose nominal values are given in Fig. 1 and are assumed to have Gaussian distribution with standard deviation $\pm 20\%$. Smolyak grids based on the Hermite Genz-Keister quadrature [16] have been used leading to an isotropic sparse grid of 13 points.

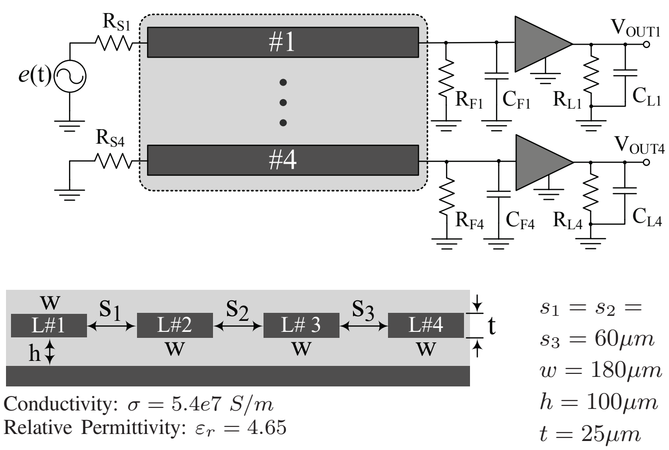

Fig. 1. A stable active circuit consisting of four coupled transmission lines and four amplifiers (Top). Cross section of the transmission lines (Bottom).

The statistical analysis of the output response using the proposed methodology was performed with $M = 3$. Fig. 2 shows the *mean* and the 3σ tolerances of the output voltage at the far end of line 1 based on 2000 MC simulations using the original model and based on the proposed technique. A good agreement between the results obtained using the two approaches is achieved.

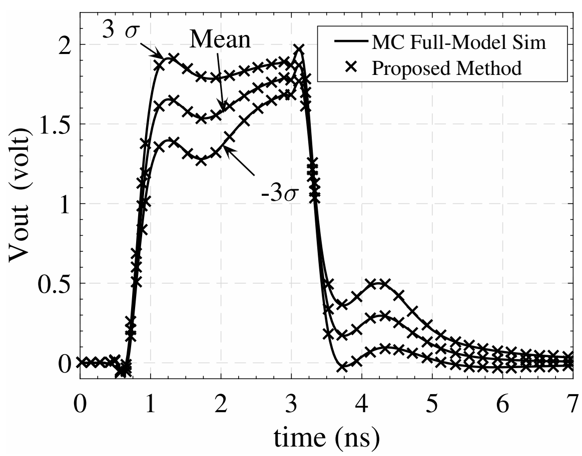

Fig. 2. Time-domain statistical analysis for the output voltage at the far end of line 1.

Table I compares these two approaches in terms of model size and CPU time. The CPU times reported here correspond to a PC platform with 4GB RAM and 1.9GHz Intel processor, executed in Matlab® (R2016a) environment. For this example, using the proposed method resulted in about 2135 CPU-time speedup ratio.

TABLE I
CPU TIME AND SIZE COMPARISON

	Size	Time (s)
MC Full-Model Simulation	1222	5316.4
Proposed Method	36	2.49

IV. CONCLUSION

An efficient method has been presented for time-domain variability analysis of large active circuits with multiple stochastic parameters. The three main key features of this method have been discussed. Numerical results have confirmed the efficiency and accuracy of the proposed technique in comparison with the standard Monte Carlo approach.

V. ACKNOWLEDGMENTS

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