# Anisotropic Formulation of Hyperbolic Polynomial Chaos Expansion for High-Dimensional Variability Analysis of Nonlinear Circuits

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*Abstract* — **In this paper, a new polynomial chaos (PC) approach for the fast variability analysis of high speed nonlinear circuits is presented. The key feature of this work is the development of an alternative anisotropic hyperbolic scheme to intelligently truncate general PC expansions. This truncation scheme not only prunes the statistically insignificant bases arising from the high degree interactions of the random dimensions but also modulates the maximum degree of expansion along each dimension based on the contribution of that dimension to the response surface. The proposed approach results in a substantially sparser PC expansion for marginal loss of accuracy.**

***Keywords*** — **Anisotropic expansion, hyperbolic polynomial chaos, microwave networks, statistical analysis, variability analysis**

## I. Introduction

One of the most critical aspect of modern computer aided design is studying the effect of manufacturing process variations and unpredictable operating conditions on the performance of microwave and millimeter-wave electronic circuits. In this context, the generalized polynomial chaos (PC) theory has emerged as a highly robust and versatile approach for the variability analysis of high speed circuits and electromagnetic (EM) systems [1]-[8]. Typically, PC approaches approximate the variability propagated from the input random dimensions of the circuit network to the responses as a linear combination of polynomial basis functions. These basis functions represent a complete set of orthogonal bases in the Hilbert space where the orthogonal property is described with respect to the joint probability distribution of the input random dimensions [9]. The coefficients of this expansion form the new unknowns of the network and can be evaluated using either intrusive stochastic Galerkin (SG) approach [1]-[3] or the more popular non-intrusive sampling based approaches [4]-[8]. Once the coefficients have been evaluated, PC expansions form metamodels of the network response parameterized in the random space which can be probed in an analytic manner. Despite the above advantages of PC approaches, it is noted that the number of coefficients to be evaluated increases drastically with the number of random dimensions. This phenomenon is called 'curse of dimensionality' and translates to an exorbitant number of deterministic simulations of the network when using non-intrusive approaches [9].

This paper presents a new and more efficient approach to construct PC expansions for variability analysis of nonlinear high speed circuits. This approach is based on a hyperbolic scheme to truncate PC expansions. This scheme automatically prioritizes the inclusion of those PC bases exhibiting low-degree interactions between the random dimensions at the expense of those exhibiting higher-degree interactions [4]. Given that the higher-degree interactions contribute statistically insignificant information to the expansion, their strategic exclusion allows the formulation of a sparser PC expansion at minimal loss of accuracy. Additionally, the maximum degree of expansion along each random dimension is modulated based on the contribution of that dimension on the network response surface when acting alone [6]. Such an expansion will be anisotropic by construction. The combination of the above hyperbolic and the anisotropic attributes will together ensure that the overall sparse PC expansion can be constructed far more cheaply than a full-blown PC expansion.

## II. Development of Proposed PC Approach

A general nonlinear network is considered where the uncertainty in the network is represented by $n$ mutually uncorrelated random variables $\boldsymbol{\lambda}=[\lambda_1,\lambda_2,..\lambda_n]^T$. The behavior of this network can be characterized by the stochastic modified nodal analysis (MNA) equations as

$$\mathbf{G}(\boldsymbol{\lambda})\mathbf{X}(t,\boldsymbol{\lambda})+\mathbf{C}(\boldsymbol{\lambda})\frac{d\mathbf{X}(t,\boldsymbol{\lambda})}{dt}+\mathbf{F}(\mathbf{X}(t,\boldsymbol{\lambda}))=\mathbf{B}(t) \qquad (1)$$

where **G** and **C** matrices contain the stamp of all the memoryless and memory lumped circuit elements respectively, **X** is the vector of stochastic voltage/current responses, **F** contains the stamp of nonlinear circuit elements, and **B** represents the vector of independent voltage and current sources. The main aim of PC approaches is to approximate the circuit response as

$$\mathbf{X}(t,\boldsymbol{\lambda})=\sum_{k=0}^{P}\mathbf{X}_k(t)\boldsymbol{\Phi}_k(\boldsymbol{\lambda}) \qquad (2)$$

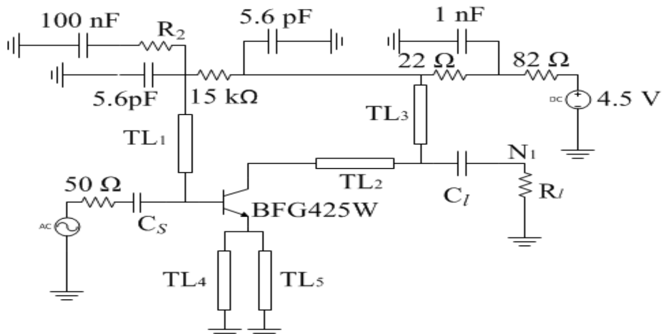



Fig. 1: Schematic of BJT LNA network.

where $\Phi_k(\boldsymbol{\lambda})$ is the $k^{th}$ degree multivariate polynomial, $\mathbf{X}_k(t)$ is the corresponding coefficient and the number of terms in the expansion of (2) is truncated to $P+1 = (n+m)!/(n!m!)$, $m$ being the maximum degree of the expansion. In the following subsections, the proposed approach to expeditiously evaluate the coefficients of (2) using an anisotropic hyperbolic scheme is described.

### A. Evaluating Anisotropic Degees of Expansion

It is intuitively obvious that the contribution of each random dimension acting alone on any response of (1) $x(t,\boldsymbol{\lambda}) \in \mathbf{X}(t,\boldsymbol{\lambda})$ can be expressed as

$$x_i(t,\lambda_i) = u(t,\boldsymbol{\lambda})\big|_{\boldsymbol{\lambda}^{(0)}\setminus\lambda_i} - x(t,\boldsymbol{\lambda}^{(0)}) \qquad (3)$$

where the notation $\boldsymbol{\lambda}^{(0)} \setminus \lambda_i$ represents the vector where all component of $\boldsymbol{\lambda}$ except $\lambda_i$ is set to 0 and $\boldsymbol{\lambda}^{(0)} = \mathbf{0}$. The contribution of (2) can now be modeled as a unidimensional (1D) PC expansions as

$$x_i(t,\lambda_i) \approx \sum_{k=1}^{m_i} x_i^{(k)}(t)\phi_k(\lambda_i) \qquad (4)$$

where $x_i^{(k)}(t)$ represents the $k^{th}$ coefficient and $\phi_k$ is the corresponding 1D basis chosen. In (4), $m_i$ is the maximum degree of the PC expansion along the arbitrary $i^{th}$ dimension. This degree is so chosen such that the $L_2$ error norm of the approximation of (4) falls below a prescribed tolerance. This approach to set the anisotropic degrees of expansion is possible since (3) separates the contribution of each random dimension $\lambda_i$ from the others. The coefficients of (4) are evaluated nonintrusively using (3) and the pseudo-spectral collocation method of [9].

### B. Anistropic Formulation of Hyperbolic PC

It is well known that a $k^{th}$ multidimensional basis can be expressed as a product of one dimensional bases as [9]

$$\Phi_k = \prod_{i=1}^{n} \phi_{d_i}(\lambda_i) \qquad (5)$$

where the subscript $d_i$ refers to the 1D PC degree of the basis $\phi_i$. By defining the new vector $d = [d_1, d_2,\ldots, d_n]$ it is pointed out that the conventional linear truncation of an isotropic PC expansion of maximum degree $m$, such as (2), includes only those multidimensional bases which satisfy the following restriction on the $L_1$ norm of $d$

$$\|d\|_1 = d_1 + d_2 + \ldots + d_n \le m \qquad (6)$$

One consequence of the linear truncation scheme of (6) is that the number of unknown coefficients in (2) scales in a polynomial manner as $O(P+1) \approx O(n^m)$ . Thus, for high-dimensional problems, this scalability may become intractable.

On the other hand, the proposed methodology is based on the knowledge that not all $P+1$ bases of (2) has equal impact on the statistics of $x(t,\boldsymbol{\lambda})$. This knowledge is supported by the sparsity of effects principle which states that the low-degree interactions between the random dimensions are statistically more significant than the higher-degree interactions [10]. Thus, it is possible to exclude the high-dimensional bases of (2) to achieve a sparser PC expansion with negligible loss of accuracy. For this purpose, the following hyperbolic truncation scheme based on the $u^{th}$ norm of the vector $d$ is employed

$$\|d\|_u = \left(d_1^{\,u} + d_2^{\,u} + \ldots + d_n^{\,u}\right)^{1/u} \le m \qquad (7)$$

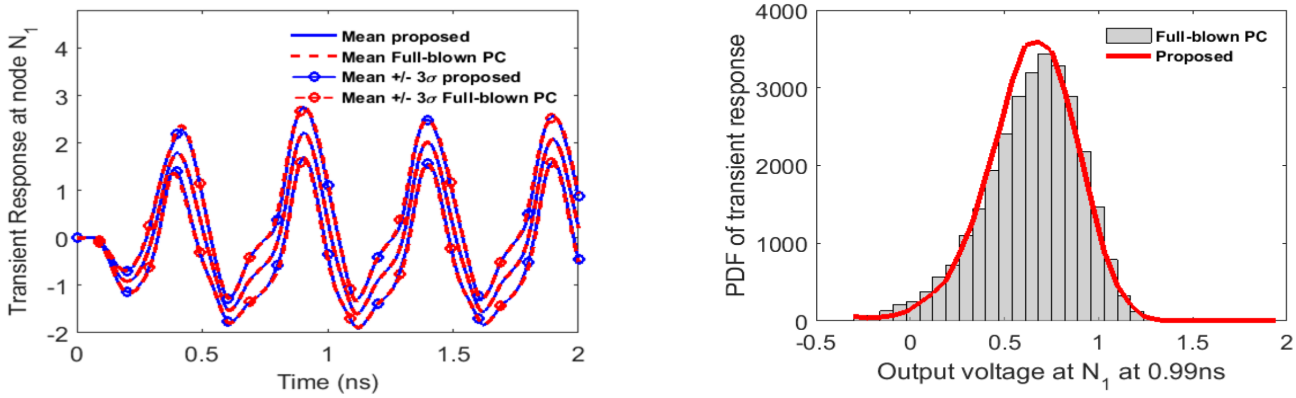
where $u$ is called the hyperbolic factor. The main feature of the hyperbolic truncation scheme of (7) is that due to the non-zero curvature of the hyperbolic function, the inclusion of the bases involving low-degree interactions between the random dimensions are automatically prioritized over their high-degree counterparts [4]. As a result, this truncation scheme results in a far sparser expansion than that of (2) and is hereafter referred to as the hyperbolic PC expansion (HPCE). However, the truncation scheme of (7) is still isotropic in nature. By adding the constraints

TABLE I
CHARACTERISTICS OF RANDOM VARIABLES OF LNA NETWORK OF FIG. 1

| No. | Random Variable | Mean | % Relative SD |
|---|---|---|---|
| 1 | $w_1$ (Width of $TL_1$) | 0.2 mm | 10 |
| 2 | $w_2$ (Width of $TL_2$) | 0.25 mm | 20 |
| 3 | $w_3$ (Width of $TL_3$) | 0.3 mm | 10 |
| 4 | $w_4$ (Width of $TL_4$) | 0.7 mm | 20 |
| 5 | $w_5$ (Width of $TL_5$) | 0.9 mm | 20 |
| 6 | $B_f$ (BJT current gain) | 145 | 30 |
| 7 | $C_{js}$ (substrate4 capacitance) | 667.5 fF | 10 |
| 8 | $R_l$ (load resistance) | 50 Ω | 10 |
| 9 | $C_l$ (load capacitance) | 2.7 pF | 20 |
| 10 | $C_s$ (source capacitance) | 4.7 pF | 20 |

TABLE II
DEGREE OF ANISOTROPIC EXPANSION ALONG RANDOM DIMENSIONS

| Dimension Number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Degree of expansion ($m_i$) | 2 | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 4 | 1 |




(a) (b)

Fig. 2: Statistics of the LNA network of Fig. 1 evaluated using conventional linear regression [5] and the proposed anisotropic HPCE. (a) Statistical analysis of the transient response at node $N_1$. (b) Probability distribution function of transient response at node $N_1$ at $t = 0.99$ ns.

$$d_1 \le m_1;\ d_2 \le m_2;\ \ldots;\ d_n \le m_n \tag{8}$$

the truncation scheme of (7) becomes anisotropic in nature [6]. This anisotropic attribute enables additional basis reduction from the isotropic rendering of the HPCE. It is emphasized that since the different degrees of (8) has been deemed adequate to capture the impact of the corresponding random dimension acting alone (see (4)), the loss in accuracy due to this extra anisotropic feature is marginal. The overall sparse PC expansion arising from the anisotropic HPCE can be formulated as

$$x(t,\boldsymbol{\lambda}) \approx \sum_{j=0}^{Q} x_j(t)\psi_j(\boldsymbol{\lambda}) \tag{9}$$

where $Q << P$. The sparse coefficients of (9) can be extracted using any non-intrusive approach, although in this paper the linear regression approach is chosen [5].

## III. Numerical Result and Discussion

In order to validate the proposed approach, the LNA network of Fig. 1 is considered. The input signal is a sinusoid with an amplitude of 1V and frequency of 2 GHz. The uncertainty in the circuit is introduced via $n = 10$ normally distributed random variables described in Table I. The lengths of the transmission lines $TL_1$, $TL_2$, $TL_3$, $TL_4$ and $TL_5$ in Fig. 1 are 8.9 mm, 3.9 mm, 6.6 mm, 3.0 mm, and 3.0 mm respectively. The transmission lines are copper microstrip traces with thickness 2 µm located on top of a dielectric plate of thickness 0.5 mm and relative permittivity of 4.6.

For this example, both the full-blown PC expansion and the anisotropic HPCE approaches are implemented. The maximum degree of expansion required is $m = 4$. For the full-blown PC approach, a total of $P$+1 = 1001 basis terms, or in other words, a total of 1001 full model SPICE simulations of the network of Fig. 1 are required. On the other hand, the proposed anisotropic HPCE approach used $u = 0.79$ with the anisotropic degrees of expansion listed in Table II. Overall, the proposed approach required only 284 SPICE simulations. Thus, for this example, the proposed approach achieves a CPU speedup of roughly 3.5 times over the full-blown approach. As expected the accuracy of the proposed approach exhibits good agreement with that of the full-blown approach as shown in Fig. 2.

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