

# Reduced Order Modeling in FDTD with Provable Stability beyond the CFL Limit

Xinyue Zhang, Fadime Bekmambetova, and Piero Triverio

The Edward S. Rogers Sr. Department of Electrical and Computer Engineering

University of Toronto, 10 King's College Road, Toronto, ON M5S3G4, Canada.

Email: xinyue.zhang.zhang@mail.utoronto.ca, fadime.bekmambetova@mail.utoronto.ca, piero.triverio@utoronto.ca

**Abstract**—The Finite-Difference Time-Domain (FDTD) method is widely used in signal and power integrity, applied electromagnetism, and physics. Unfortunately, its computational efficiency can be severely degraded for multiscale problems, where small and large features coexist. This scenario is common in signal and power integrity, because of the large aspect ratio of interconnects and power/ground planes. In this paper, we show how multiscale FDTD simulations can be accelerated with model order reduction. A detailed model for complex objects is first generated using a fine FDTD grid. The model is then compressed with model order reduction, and embedded into a main coarse grid. During this process, the stability limit of the reduced model can be also extended, enabling the use of a larger time step in the whole domain. Using a passivity argument, we are able to systematically guarantee the stability of the resulting scheme, which is a main novelty with respect to previous works. A numerical example with two reduced models shows the potential of the proposed ideas.

**Index Terms**—Finite-Difference Time-Domain, Model Order Reduction, CFL Limit Extension, Stability

## I. INTRODUCTION

The Finite-Difference Time-Domain (FDTD) method solves Maxwell's equations recursively, avoiding the solution of linear systems which can be expensive for large problems [1]. Among many applications, FDTD is also used to analyze interconnects, boards and chip packages in order to investigate signal and power integrity issues [2]. Unfortunately, the recursive nature of FDTD imposes a restriction on time step  $\Delta t$ , known as Courant-Friedrichs-Lewy (CFL) stability limit, which in 2D reads

$$\Delta t < \frac{1}{c\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}}, \quad (1)$$

where  $\Delta x$  and  $\Delta y$  are FDTD cell dimensions, and  $c$  is the wave velocity in the medium. The CFL limit is dictated by the size of the smallest cell, which can lead to very long simulations for multiscale problems. Moreover, with fine meshes, more fields need to be updated, which further increases computational cost and memory consumption.

Many approaches have been pursued to improve FDTD efficiency in multiscale scenarios. In spatial subgridding, one refines the mesh only locally, to achieve better resolution where necessary, but the maximum time step remains constrained by the CFL limit of the fine mesh [3]. While different time steps can be used in the coarse and fine grids, proving

the stability of the resulting scheme can be a formidable task [4]. Implicit integration can be used to achieve stability for any timestep, but this comes at an extra cost, even when model order reduction (MOR) is applied [5]. An alternative approach to overcome the CFL barrier while maintaining the explicit nature of FDTD is CFL limit extension, where one eliminates the unstable components of the FDTD solution with a perturbation approach [6], [7] or with low-pass filtering at runtime [8]. A drawback of these methods is that they must be applied to the whole problem, which limits scalability.

In this paper, we show that one can control and extend the CFL limit of a multiscale simulation by operating only on the subregions where fine geometrical details are present. First, these subregions are discretized with a fine FDTD mesh, in order to capture their complex geometry. The obtained models are compressed with MOR, which provides a first level of complexity reduction. Then, the reduced model is perturbed in order to extend its stability limit, which is otherwise constrained by the fine resolution. The CFL limit extension provides a second level of complexity reduction. Finally, the reduced model is embedded into the surrounding coarse grid. Through a passivity argument [9], we prove that the obtained scheme with embedded models is stable up to the timestep chosen during the CFL limit extension. Guaranteed stability is a key novelty with respect to previous works [10]. A numerical test confirms the stability of the proposed technique, and its potential for accelerating multiscale FDTD simulations.

## II. PROPOSED FORMULATION

The proposed method is presented in 2D, by considering a transverse-electric field with  $E_x$ ,  $E_y$  and  $H_z$  components. In many multiscale problems, the fine geometrical features are present only in selected regions, that we discretize with a fine FDTD grid. A coarse grid is instead used elsewhere, as shown in the left panel of Fig. 1. Without loss of generality, we assume that the interface between the coarse and fine grids is on the primary FDTD grid, where the electric field is sampled.

### A. Discrete-Time Dynamical Model for FDTD Meshes

In [9], we show that the FDTD equations for a rectangular region can be cast into a convenient matrix form in order to obtain a complete dynamical model for the electromagnetic field in the region. For all electric and magnetic nodes that fall

strictly inside the region, a standard FDTD update equation is used. For the E nodes on the coarse-fine interface, a standard FDTD equation would involve some magnetic samples beyond the region boundaries. In order to avoid this, and obtain a self-contained model for the region suitable for MOR, the magnetic field on the boundary is also sampled. These additional samples, known as hanging variables, will facilitate the connection of the fine-grid reduced model to the main grid. This process leads to a discrete-time dynamical model for the fine grid [9]

$$(\mathbf{R} + \mathbf{F})\mathbf{x}^{n+1} = (\mathbf{R} - \mathbf{F})\mathbf{x}^n + \mathbf{B}\mathbf{u}^{n+\frac{1}{2}}, \quad (2a)$$

$$\mathbf{y}^n = \mathbf{L}^T \mathbf{x}^n, \quad (2b)$$

where

$$\mathbf{x}^n = \begin{bmatrix} \mathbf{E}_x^n \\ \mathbf{E}_y^n \\ \mathbf{H}_z^{n-\frac{1}{2}} \end{bmatrix} \quad (3)$$

is the state vector, which contains the electric and magnetic field samples at every node. The input and output vectors are given by

$$\mathbf{u}^{n+\frac{1}{2}} = \begin{bmatrix} \mathbf{H}_S^{n+\frac{1}{2}} \\ \mathbf{H}_N^{n+\frac{1}{2}} \\ \mathbf{H}_W^{n+\frac{1}{2}} \\ \mathbf{H}_E^{n+\frac{1}{2}} \end{bmatrix}, \quad \mathbf{y}^n = \begin{bmatrix} \mathbf{E}_S^n \\ \mathbf{E}_N^n \\ \mathbf{E}_W^n \\ \mathbf{E}_E^n \end{bmatrix}. \quad (4)$$

Input vector  $\mathbf{u}^{n+\frac{1}{2}}$  contains all hanging variables, that is all magnetic samples introduced on the South, North, West and East boundaries of the fine grid. Output vector  $\mathbf{y}^n$  contains the electric field at the same locations. The coefficient matrices in (2a)-(2b) depend on the permittivity, permeability and conductivity in the region [9], as well as  $\Delta x$ ,  $\Delta y$  and  $\Delta t$ .

### B. Model Order Reduction

The fine grid model (2a)-(2b) is compressed using the SPRIM method [11], a structure-preserving MOR technique well suited to maintain the structure of FDTD equations [6]. SPRIM produces a projection matrix  $\mathbf{V}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_3 \end{bmatrix}, \quad (5)$$

which has three diagonal blocks, in accordance with the structure of (3). Using (5), the full state vector  $\mathbf{x}^n$  can be approximated with a reduced state  $\tilde{\mathbf{x}}^n$  as

$$\mathbf{x}^n \approx \mathbf{V}\tilde{\mathbf{x}}^n. \quad (6)$$

By substituting (6) into (2a)-(2b), and multiplying by  $\mathbf{V}^T$  from the left side, a reduced model is obtained

$$(\tilde{\mathbf{R}} + \tilde{\mathbf{F}})\tilde{\mathbf{x}}^{n+1} = (\tilde{\mathbf{R}} - \tilde{\mathbf{F}})\tilde{\mathbf{x}}^n + \tilde{\mathbf{B}}\mathbf{u}^{n+\frac{1}{2}}, \quad (7a)$$

$$\tilde{\mathbf{y}}^n = \tilde{\mathbf{L}}^T \tilde{\mathbf{x}}^n, \quad (7b)$$

where  $\tilde{\mathbf{R}} = \mathbf{V}^T \mathbf{R} \mathbf{V}$ ,  $\tilde{\mathbf{F}} = \mathbf{V}^T \mathbf{F} \mathbf{V}$ ,  $\tilde{\mathbf{B}} = \mathbf{V}^T \mathbf{B}$  and  $\tilde{\mathbf{L}} = \mathbf{V}^T \mathbf{L}$ .

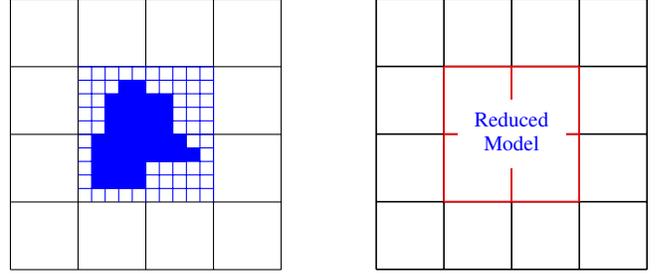


Fig. 1. Left panel: initial mesh with a fine grid used to resolve complex objects, and a coarse grid elsewhere. Right panel: final setup where the fine grid has been replaced by a reduced order model.

### C. Stability Analysis and CFL Limit Extension

Once the reduced model has been generated, it can be connected back to the coarse mesh, imposing a suitable interpolation rule to relate the fields on the two sides of the coarse-fine interface, that are sampled with different resolution. From a system theory viewpoint, the resulting scheme can be interpreted as the connection of three subsystems: the coarse grid, the reduced model, and the interpolation rule in between [9]. The stability of this complex setup can be guaranteed with a passivity argument [9]. By requiring each one of the three subsystems to be passive, the overall system formed by their connection will be passive by construction, and thus stable [12]. As discussed in [9], the coarse grid is passive for any  $\Delta t$  below its CFL limit. As interpolation rule, we use the scheme in [9], which is lossless for any timestep. The reduced model (7a)-(7b) can be shown to be passive under the CFL limit of the *fine* grid, which is more restrictive than the coarse grid's limit. As a result, the whole scheme will be stable only up to the CFL limit of the fine grid. This fact is undesirable since a refinement of the fine grid imposes a finer timestep throughout the entire domain.

We overcome this issue with a perturbation approach, aimed at making the fine model passive beyond its native CFL limit. Passivity conditions [9] require the  $\tilde{\mathbf{R}}$  matrix in (7a) to be positive definite. This condition can be shown to be a generalization of the CFL condition, and is violated for any  $\Delta t$  beyond the CFL limit of the fine grid. The positive definiteness of  $\tilde{\mathbf{R}}$  can be restored by perturbing some coefficients in (7a), in a similar way as [6]. An extension of the CFL limit by 3-4 times can be typically achieved with marginal impact on accuracy. Making the fine grid model passive for a higher  $\Delta t$  extends the stability limit of the whole scheme enabling the use of a larger time step over the *entire* computational domain. Owing to the passivity argument, the stability of the resulting scheme is provable for any placement and number of embedded reduced models, which is a novel result. Moreover, it is remarkable that the stability of the whole setup can be precisely controlled by operating solely on the fine region, which is the ultimate cause for the CFL limit reduction. This feature makes the proposed approach scalable to very large problems, since no operation is required on the host coarse grid, which is updated with conventional FDTD equations.

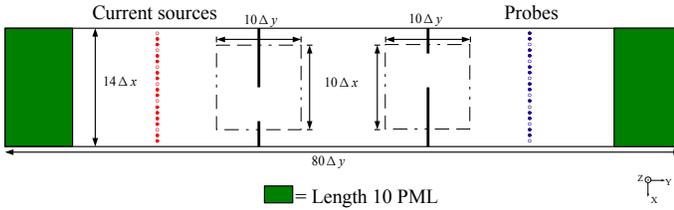


Fig. 2. Waveguide geometry. The boxes indicate the regions represented with a reduced model.

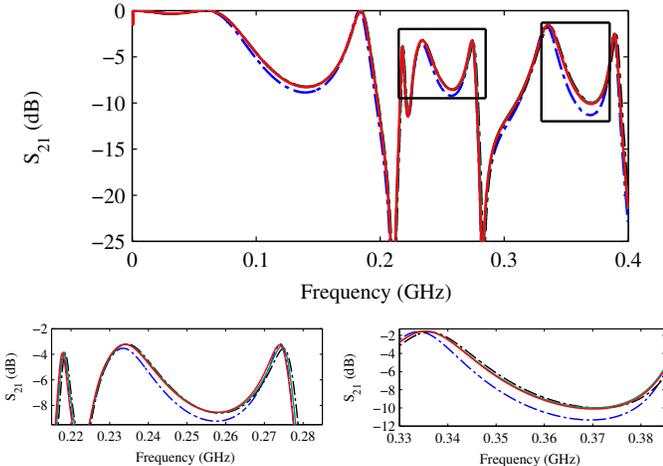


Fig. 3. Frequency response of the waveguide obtained with coarse grid everywhere (— — —), fine grid everywhere (— — —), subgridding (— — —) and proposed method (— — —).

### III. NUMERICAL EXAMPLE

The proposed method was tested on a waveguide structure with two irises, shown in Fig. 2. The waveguide is 0.7 m long and 4 m wide, and was discretized with a coarse grid with  $\Delta x = \Delta y = 5$  cm. Two very thin perfect electric conductor (PEC) irises of length 0.2 m are placed along the waveguide 1 m apart. The waveguide is terminated on perfectly matched layers and excited by a Gaussian pulse with 0.4 GHz bandwidth. The coarse mesh was refined by 3 times in the two regions around the apertures, because of the small thickness of the irises and the local irregularity of the fields. A reduced model was then generated for these regions using the proposed method, and its CFL limit was extended by two times to double the timestep that can be used to update the whole domain. Figure 3 shows the frequency response of the waveguide computed with the proposed method, subgridding [9], an all-coarse FDTD mesh, and an all-fine FDTD mesh. All methods are in good agreement with the reference all-fine simulation, except for the coarse grid, that leads to visible deviations. The execution times for the different methods are presented in Table I. The presence of two refined regions makes subgridding only barely more efficient than an all-fine FDTD. The proposed method accelerates the whole simulation by 2.37X. This result confirms the merit of the proposed idea, that is being currently extended to larger problems and three dimensions.

TABLE I

EXECUTION TIME FOR THE DIFFERENT METHODS. FOR THE PROPOSED METHOD, THE MOR TIME IS INCLUDED IN THE RUNTIME.

Case	CFL number	Runtime	Speed-up
All Coarse	0.99	28.9 s	-
All Fine	0.99	281.2 s	-
Subgridding	0.99	222.0 s	1.26X
Proposed Method	1.98	118.5 s	2.37X

### IV. CONCLUSION

We proposed a systematic approach to embed reduced order models into a main FDTD grid, in order to model fine geometrical features in an efficient way. Complex objects are first modelled with a fine FDTD mesh. The model is then compressed with model order reduction, and instantiated into the main coarse grid. The stability of the resulting scheme can be rigorously proved with a passivity argument. Using a perturbation approach, it is also possible to enhance the stability properties of the reduced model, and make the whole scheme stable for a timestep beyond the native stability limit. A numerical example confirmed the stability and efficiency of the method.

### REFERENCES

- [1] K. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propag.*, vol. 14, no. 3, pp. 302–307, 1966.
- [2] M. Yi, M. Ha, Z. Qian, A. Aydiner, and M. Swaminathan, "Skin-effect-incorporated transient simulation using the Laguerre-FDTD scheme," *IEEE Trans. Microw. Theory Tech.*, vol. 61, no. 12, Dec 2013.
- [3] M. Okoniewski, E. Okoniewska, and M. Stuchly, "Three-dimensional subgridding algorithm for FDTD," *IEEE Trans. Antennas Propag.*, vol. 45, no. 3, pp. 422–429, 1997.
- [4] K. Xiao, D. J. Pommerenke, and J. L. Drewniak, "A three-dimensional FDTD subgridding algorithm with separated temporal and spatial interfaces and related stability analysis," *IEEE Trans. Antennas Propag.*, vol. 55, no. 7, pp. 1981–1990, 2007.
- [5] B. Denecker, F. Olyslager, L. Knockaert, and D. De Zutter, "Generation of FDTD subcell equations by means of reduced order modeling," *IEEE Trans. Antennas Propag.*, vol. 51, no. 8, pp. 1806–1817, 2003.
- [6] X. Li, C. D. Sarris, and P. Triverio, "Structure-Preserving Reduction of Finite-Difference Time-Domain Equations with Controllable Stability Beyond the CFL Limit," *IEEE Trans. Microw. Theory Tech.*, vol. 62, no. 12, pp. 3228–3238, 2014.
- [7] M. Gaffar and D. Jiao, "An Explicit and Unconditionally Stable FDTD Method for Electromagnetic Analysis," *IEEE Trans. Microw. Theory Tech.*, vol. 62, no. 11, pp. 2538–2550, 2014.
- [8] C. Chang and C. D. Sarris, "A Spatially Filtered Finite-Difference Time-Domain Scheme With Controllable Stability Beyond the CFL Limit: Theory and Applications," *IEEE Trans. Microw. Theory Tech.*, vol. 61, no. 1, pp. 351–359, Jan 2013.
- [9] F. Bekmambetova, X. Zhang, and P. Triverio, "A dissipative systems theory for FDTD with application to stability analysis and subgridding," *IEEE Trans. Antennas Propag.*, 2016, (submitted).
- [10] X. Li and P. Triverio, "Stable FDTD Simulations with Subgridding at the Time Step of the Coarse Grid: a Model Order Reduction Approach," in *IEEE MTT-S Int. Conf. on Numerical Electromagnetic and Multiphysics Modeling and Optimization*, Ottawa, Canada, August 11–14 2015.
- [11] R. W. Freund, "SPRIM: Structure-preserving reduced-order interconnect macromodeling," in *Proceedings of the 2004 IEEE/ACM International Conference on Computer-aided Design*, ser. ICCAD '04. Washington, DC, USA: IEEE Computer Society, 2004, pp. 80–87.
- [12] P. Triverio, S. Grivet-Talocia, M. S. Nakhla, F. Canavero, R. Achar, "Stability, causality, and passivity in electrical interconnect models," *IEEE Trans. Adv. Packag.*, vol. 30, no. 4, pp. 795–808, 2007.