

Stability Analysis for Semi-Implicit LIM Algorithm

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Abstract - The latency insertion method (LIM) has been demonstrated as an optimum algorithm for the transient simulation large networks. However, its stability is not unconditional. This paper reviews the semi-implicit formulation of LIM and examines the stability properties of the formulation using a block matrix approach. Stability conditions are derived and examples are used to validate the results.

I. INTRODUCTION

The simulation of large networks plays an important role in integrated circuits design. Such simulation must be performed in a computationally efficient manner in order to reduce designer turn-around time. The LIM algorithm has established itself as an attractive method for the fast simulation of large networks [1]. It takes advantage of inherent or artificial latency in a network to generate a leapfrog algorithm similar to the Yee algorithm used in FDTD [2]. However, in order to obtain full efficiency from the method, stability must be insured by properly choosing the time step. The stability properties of LIM have been examined in previous work [3]. In this paper, we propose a stability analysis based on a block matrix semi-implicit formulation.

II. BASIC FORMULATION

Complex networks found in power distribution networks can be described as multidimensional interconnects consisting of series and shunt elements. These elements can be represented as combinations of sources, and reactive and resistive elements. In the basic LIM algorithm, they are assumed to be linear. First, the time variable is discretized, next the voltage and current quantities are collocated in half time steps to generate sequences of the form $V^{n-1/2}$, $V^{n+1/2}$, $V^{n+3/2}$ for voltages and I^n , I^{n+1} , I^{n+2} for currents. Referring to Figure 1, each branch is represented as a combination of a voltage source, an inductor and a resistor in series. If no inductance exists in a given branch, then value of the inductor can be made relatively small. Current I_{ij} in each branch is assumed to be directed from node i at voltage V_i to node j at potential V_j . Each node is modeled as a parallel combination of a current source, a conductance and a capacitor to ground.

If no capacitance to ground exists from that node, a relatively small capacitor is introduced to generate the latency.

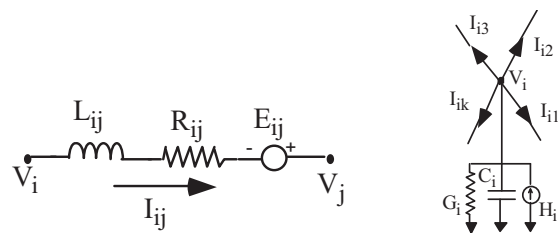


Figure 1. Branch and node representation for LIM formulation.

Applying Kirchhoff's voltage law to each branch yields, the semi-implicit LIM formulation yields [3]

$$V_i^{n+1/2} - V_j^{n+1/2} = L_{ij} \left(\frac{I_{ij}^{n+1} - I_{ij}^n}{\Delta t} \right) + \frac{R_{ij}}{2} (I_{ij}^{n+1} + I_{ij}^n) - E_{ij}^{n+1/2} \quad (1)$$

where L_{ij} , R_{ij} and E_{ij} are the inductance, resistance and independent voltage source respectively. Kirchhoff's current law formulated in a semi-implicit form at each node reads

$$C_i \left(\frac{V_i^{n+1/2} - V_i^{n-1/2}}{\Delta t} \right) + \frac{G_i}{2} (V_i^{n+1/2} + V_i^{n-1/2}) - H_{ij}^n = - \sum_{k=1}^{M_i} I_{ik}^n \quad (2)$$

where C_i , G_i and H_i are the capacitance, conductance and independent current source respectively. M_i is the number of branches connected to node i (excluding connections to ground). At each time step, these operations are performed over all N_b branches and all N_n nodes in order to update all the current and voltage quantities respectively. Altogether, $N_n(N_n + N_b)$ operations are performed to obtain $N_n(N_n + N_b)$ values yielding hence an optimally efficient algorithm.

III. SEMI-IMPLICIT BLOCK LIM FORMULATION

The update equations in (1) and (2) are known to be conditionally stable. The time step must satisfy certain constraints

to guarantee stability. Equation (3) can be written in a vector-matrix formulation as:

$$\mathbf{C} \left(\frac{\mathbf{v}^{n+1/2} - \mathbf{v}^{n-1/2}}{\Delta t} \right) + \frac{1}{2} \mathbf{G} (\mathbf{v}^{n+1/2} + \mathbf{v}^{n-1/2}) - \mathbf{h}^n = -\mathbf{M} \mathbf{i}^n \quad (3)$$

where \mathbf{M} is the $N_n \times N_b$ incidence matrix defined as follows

$M_{qp} = 1$ if branch p is incident at node q and the current flows away from node q .

$M_{qp} = -1$ if branch p is incident at node q and the current flows into node q .

$M_{qp} = 0$ if branch p is not incident at node q .

$\mathbf{v}^{n+1/2}$ is the node voltage vector (dimension N_b) representing the voltages at the nodes at $n+1/2$, \mathbf{i}^n is the current vector (dimension N_n) representing the currents in all the branches at time n . \mathbf{G} and \mathbf{C} are the conductance and capacitance matrices respectively. They are of dimension N_n by N_n . Solving for $\mathbf{v}^{n+1/2}$, we get:

$$\mathbf{v}^{n+1/2} = \left(\frac{\mathbf{C}}{\Delta t} + \frac{\mathbf{G}}{2} \right)^{-1} \left[\left(\frac{\mathbf{C}}{\Delta t} - \frac{\mathbf{G}}{2} \right) \mathbf{v}^{n-1/2} + \mathbf{h}^n - \mathbf{M} \mathbf{i}^n \right] \quad (4)$$

Similarly, the branch equation can be arranged in a matrix form to yield

$$\mathbf{M}^T \mathbf{v}^{n+1/2} = \frac{\mathbf{L}}{\Delta t} (\mathbf{i}^{n+1} - \mathbf{i}^n) + \frac{\mathbf{R}}{2} (\mathbf{i}^{n+1} + \mathbf{i}^n) - \mathbf{e}^{n+1/2} \quad (5)$$

\mathbf{R} and \mathbf{L} are the resistance and inductance matrices respectively. They are of dimension N_b by N_b . Solving for \mathbf{i}^n , we get:

$$\mathbf{i}^{n+1} = \left(\frac{\mathbf{L}}{\Delta t} + \frac{\mathbf{R}}{2} \right)^{-1} \left[\left(\frac{\mathbf{L}}{\Delta t} - \frac{\mathbf{R}}{2} \right) \mathbf{i}^n + \mathbf{e}^{n+1/2} + \mathbf{M}^T \mathbf{v}^{n+1/2} \right] \quad (6)$$

which are the update equations.

IV. STABILITY - AMPLIFICATION MATRIX

If we assume that the sources are zero, then equations (4) and (6) can be combined together in a block matrix form to read:

$$\begin{bmatrix} \mathbf{v}^{n+1/2} \\ \mathbf{i}^{n+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{v}^{n-1/2} \\ \mathbf{i}^n \end{bmatrix} \quad (7)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_+ \mathbf{P}_- & -\mathbf{P}_+ \mathbf{M} \\ \mathbf{Q}_+ \mathbf{M}^T \mathbf{P}_+ \mathbf{P}_- & \mathbf{Q}_+ \mathbf{Q}_- - \mathbf{Q}_+ \mathbf{M}^T \mathbf{P}_+ \mathbf{M} \end{bmatrix} \quad (8)$$

in which we have defined

$$\mathbf{P}_+ = \left(\frac{\mathbf{C}}{\Delta t} + \frac{\mathbf{G}}{2} \right)^{-1} \quad \mathbf{P}_- = \left(\frac{\mathbf{C}}{\Delta t} - \frac{\mathbf{G}}{2} \right) \quad (9a)$$

$$\mathbf{Q}_+ = \left(\frac{\mathbf{L}}{\Delta t} + \frac{\mathbf{R}}{2} \right)^{-1} \quad \mathbf{Q}_- = \left(\frac{\mathbf{L}}{\Delta t} - \frac{\mathbf{R}}{2} \right) \quad (9b)$$

\mathbf{A} is the amplification matrix. It is obvious if multiplication by the matrix \mathbf{A} lead to larger values, growth will occur leading to unstable behavior. Therefore, stability entails that the time step Δt be chosen such that the spectral radius of the amplification matrix be less than unity.

V. SINGLE-CELL STABILITY CONDITION

In order to apply the method, we consider the circuit shown in Figure 2. This circuit has one branch and two nodes.

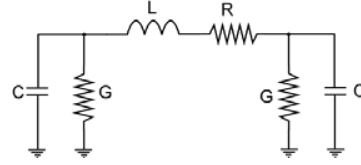


Figure 2. Single-cell circuit used for testing stability.

Consequently, the amplification matrix for this single-cell circuit is described by:

$$\mathbf{A} = \begin{pmatrix} \Gamma_y & 0 & -T_y \\ 0 & \Gamma_y & T_y \\ T_z \Gamma_y & T_z \Gamma_y & \Gamma_z - 2T_z T_y \end{pmatrix} \quad (10)$$

in which we have made the definitions

$$T_y = \frac{2h}{2C + Gh} \quad \Gamma_y = \frac{2C - Gh}{2C + Gh}, \quad (11a)$$

$$T_z = \frac{2h}{2L + Rh}, \quad \text{and} \quad \Gamma_z = \frac{2L - Rh}{2L + Rh} \quad (11b)$$

where h is the time step ($h = \Delta t$). In order to solve for the eigenvalues, λ we set $|\mathbf{A} - \lambda \mathbf{I}| = 0$ which leads to

$$\lambda^2 + \lambda[2T_z T_y - \Gamma_y - \Gamma_z] + \Gamma_y \Gamma_z = 0 \quad (12)$$

Again, define

$$D = 2T_z T_y - \Gamma_y - \Gamma_z \quad \text{and} \quad W = \Gamma_y \Gamma_z \quad (13)$$

The quadratic equation leads to

$$\lambda^2 + \lambda D + W = 0 \quad (14)$$

which gives the roots

$$\lambda = \frac{-D \pm \sqrt{D^2 - 4W}}{2} \quad (15)$$

For stability, we want $|\lambda| \leq 1$. In the special case when $G=0$, this condition becomes $h \leq \sqrt{2LC}$. Notice that the condition does not depend on R .

VI. TESTS AND RESULTS

In order to test the proposed method, the simple circuit shown in Figure 2 is analyzed. The circuit has $R=100 \Omega$, $L=1 \text{ nH}$, $C=2 \text{ pF}$ and $G=0$. From the derivation above, it was found that the maximum allowable time step was $\Delta t_{\max} = \sqrt{2LC} = 0.0632456$. Figure 3 shows simulation results for time steps slightly above and below the maximum time step, resulting in unstable and stable response respectively. These results validate the stability condition.

In the analysis that we performed, the total capacitance in the cell is $2C$. If the total capacitance per unit cell is C , we divide the capacitance into two halves and make the transformation $C \rightarrow C/2$. The time step condition then becomes $\Delta t_{\max} = \sqrt{LC}$ which is the well-known semi-implicit condition from FDTD [3].

CONCLUSION

This paper presented a semi-implicit block-matrix formulation of the LIM algorithm. From the formulation, an amplification matrix was derived and conditions stability was determined. In the special case where the conductance to ground is null, it is found that for a single cell network the stability condition on the time step does not depend on the value of the resistance. Simulations on a simple circuit were used to validate these results.

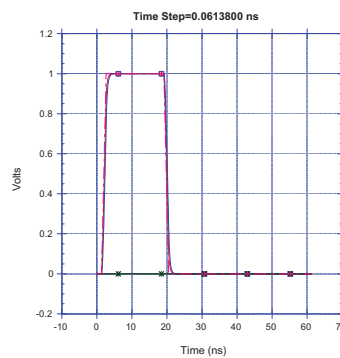
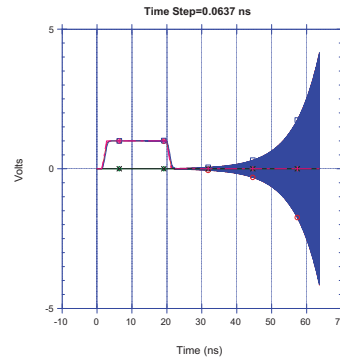


Figure 3. Simulated LIM voltage responses for $\Delta t > \Delta t_{\max}$ (top) and $\Delta t < \Delta t_{\max}$ (bottom).

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