

Fast Broadband Macromodeling Technique of Sampled Time/Frequency Data using z-domain Vector-fitting Method

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Abstract—The vector-fitting algorithm has been used as the main macromodeling tool for approximating frequency domain responses of complex interconnects and electrical packages [5]. In this paper, a new methodology is proposed to fit transfer functions of frequency response data obtained from numerical electromagnetic simulation or measured frequency-domain or time-domain response data. This new method, z-domain vector-fitting (ZDVF), is a formulation of vector-fitting method in the z domain; it has an advantage of faster convergence and better numerical stability compared to the s-domain vector-fitting method (VF). The fast convergence of the method reduces the overall macromodel generation time. The accuracy, numerical stability and convergence speed of VF and ZDVF are compared. Examples are provided to demonstrate the advantage of the ZDVF.

I. INTRODUCTION

An accurate simulation of interconnects requires the transient analysis of nonlinear driver and receiver, and frequency-dependent transmission-line systems. Consequently, the most efficient method of analyzing frequency-dependent component using conventional time-domain circuit simulation method is to generate rational approximation in order to express the interconnect governing equations using ordinary differential equation.

Several methods have been proposed to generate rational representations of distribute systems. These methods include asymptotic waveform evaluation and complex frequency hopping methods and its variant that are based on moment matching at single or multiple points and other methods that are based on Krylov subspace techniques such as Lanczos and Arnoldi methods [1]- [2]. Robust rational interpolation that based on Hilbert transform of electrical parameters of passive networks or Kramers-Krönig relations between real and imaginary parts of material properties, have also been used to generate rational approximations of frequency-dependent parameters [3], [4]. However, the efficiency, accuracy, or stability of these approximation techniques has been compromised in an effort to generate high-order approximations of frequency-dependent parameters over a wide frequency range.

The vector-fitting algorithm has emerged as a method of choice in the generation of high-order rational functions over wide frequency ranges [5]. The method is an iterative technique based on pole relocation technique. It is shown that the iterative pole-relocation technique is the reformulation of

Sanathanan-Koerner (SK) iteration using partial fractions basis [7]. The method has two main advantages, namely, numerical stability and convergence. It does not suffer from numerical problem when approximating high-order system over a wide frequency range [9]. It arrives at the optimal solution by solving two linear solutions with few iterations.

However, the accuracy and convergence of the vector fitting method depends strongly on the number and the location of the starting poles. The standard methods for selecting the orders and initial poles are in the most part heuristic [5]- [6]. Orthogonal vector fitting (OVF) has been introduced in [7] to reduce the numerical sensitivity of the system equations to the choice of starting poles.

In this paper, we propose a new method of approximating a rational function using a vector-fitting method in the z-domain (ZDVF). This has the advantage of faster convergence and better numerical stability compared to the s-domain vector-fitting method (VF). The fast convergence behavior of this method reduces the overall macromodel generation time. This paper is organized as follows. First, a brief discussion on how to generate the z-domain data using bilinear transformation is presented. Next, the mathematical formulation of the proposed ZDVF method is presented. Next, starting pole selection is discussed. Then, the state-space formulation for the time-domain simulation is presented. A numerical example is presented to illustrate the advantage of the ZDVF method on fast convergence and numerical stability as compared to the VF method. Finally, a conclusion is given.

II. S-TO-Z SYSTEM TRANSFORMATION

The z-domain response data can be generated using either by using bilinear transformation from s-domain to z-domain or using Inverse Fourier Transform (IFFT) to time-domain and generate z-domain response data from time-domain response data. The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \quad (1)$$

where $z = e^{j\omega}$

A. The bilinear transformation

The bilinear transformation is a linear fractional transformation given by $\beta : \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$\beta : s \rightarrow z = \frac{\alpha + s}{\alpha - s} \quad (2)$$

where $\alpha \in \mathbb{R}$ is a constant equal to twice the sampling rate. This mapping has the property that it transforms the $j\omega$ -axis in the s-plane onto the unit circle ($z = e^{j\omega}$) in the z-plane. Moreover, the left-hand side of s-plane ($\Re(s) < 0$) is mapped inside the unit circle in the z-plane and the right-hand side of s-plane ($\Re(s) > 0$) is mapped outside of the unit circle in the z-plane. thus preserves the system stability. The inverse bilinear transformation β^{-1} is given by $\beta(-\alpha/z)$.

The main advantage of using bilinear transformation over other transformations such as the impulse- and step- invariant transformations, is that it preserves the magnitude characteristics of the transfer function. This follows from the fact that the parameter α provides a one degree of freedom that can be used to make the frequency response characteristic of the z-domain system function $f(z)$ approximate those of the s-domain system function $f(s)$.

III. Z-DOMAIN VECTOR-FITTING METHOD (ZDVF)

Rational approximation in pole-residue forms is an auxiliary step in order to make the analytical calculation of inverse Laplace or Fourier transform possible. When the order of approximation and frequency range are increased, most approximation methods suffer from numerical problems, such as ill-conditioned systems, and spurious unstable poles. In order to mitigate this problem, multi-point Padé approximation and rational interpolation over partitioned frequency range in several sub ranges have been utilized. Although the accuracy and order of approximation have significantly improved through these methods, the problems do not completely disappear.

The z-domain response of a dynamic system can be constructed from the the time domain response or by using s-to-z bilinear transformation. The z-domain response $f(z)$ of any linear time-invariant passive network can be represented using rational function. For an N^{th} -order system response, the rational function $f(z)$ can be written as

$$f(z) \approx \frac{N(z)}{D(z)} \approx \frac{\sum_{n=0}^M b_n z^{-n}}{\sum_{n=0}^N a_n z^{-n}} \quad (3)$$

where $a_0 = 1$

The approximation can be expressed in terms of the sum of a partial fraction expansion form shown in (4) where c_n and a_n are the residues and poles respectively.

$$f(z) \approx \sum_{n=1}^N \frac{c_n}{1 - z^{-1} a_n} \quad (4)$$

$$\hat{f}(z) = \frac{f(z)}{z} \approx \sum_{n=1}^N \frac{c_n}{z - a_n} \quad (5)$$

Equation (5) is the response normalized by discrete frequency and is used to find the poles and the residues without losing generality. The coefficients of the above nonlinear equation is determined by decomposing the nonlinear problem into two linear problems. The first linear problem is a pole identification problem in which the poles of the system are identified from the normalized z-domain response data and an initial starting

poles. The second linear problem is to determine the residues based on the normalized z-domain response data and the new system poles determined from the pole identification linear problem.

A. Pole Identification

First, a scalar augmented function $\sigma(z)$ is defined strictly in (6) where \tilde{c}_n and \tilde{a}_n are unknown residues and known initial starting poles respectively and N is the order of approximation.

$$\sigma(z) = \sum_{n=1}^N \frac{\tilde{c}_n}{z - \tilde{a}_n} + 1 \quad (6)$$

Multiplying the normalized z-domain response $\hat{f}(z)$ by an unknown augmented function $\sigma(z)$ results in (7) and can be rewritten as (8).

$$(\sigma \hat{f})_{fit}(z) \approx \sum_{n=1}^N \frac{c_n}{z - \tilde{a}_n} \quad (7)$$

$$\sum_{n=1}^N \frac{c_n}{z - \tilde{a}_n} - \left(\sum_{n=1}^N \frac{\tilde{c}_n}{z - \tilde{a}_n} \right) \hat{f}(z) = \hat{f}(z) \quad (8)$$

Equation (8) is linear in its unknown c_n and \tilde{c}_n . Writing (8) for the sample points gives an over determined linear problem and residues c_n and \tilde{c}_n can be solved. Now $\sigma(z)$ and $(\sigma \hat{f})_{fit}(z)$ can be rewritten as a fraction of numerator and denominator polynomial as

$$(\sigma \hat{f})_{fit}(z) \approx \frac{\prod_{n=1}^N (z - v_n)}{\prod_{n=1}^N (z - \tilde{a}_n)}, \quad \sigma(z) \approx \frac{\prod_{n=1}^N (z - \tilde{v}_n)}{\prod_{n=1}^N (z - \tilde{a}_n)} \quad (9)$$

From (9), $\hat{f}(z)$ can be expressed as

$$\hat{f}(z) \approx \frac{(\sigma \hat{f})_{fit}(z)}{\sigma(z)} \approx \frac{\prod_{n=1}^N (z - v_n)}{\prod_{n=1}^N (z - \tilde{v}_n)} \quad (10)$$

From (10), the poles of $\hat{f}(z)$ become equal to the zeros of $\sigma(z)$. The zeros of $\sigma(z)$ can be determined by calculating the eigenvalues of the matrix

$$\{a_n\} = eig(A - B.C^T) \quad (11)$$

In (11), A is a diagonal matrix holding the initial poles \tilde{a}_n , B is a column vector of ones, and C^T is a row vector holding the residues \tilde{c}_n . Since the poles of the weight function (6), and augmented transfer function (7), are identical, the initial starting poles cancellation must occur in determining $\hat{f}(z)$. The stable poles are the poles inside the unit circle. If unstable poles are found outside of the unit circle, these poles can either be forced to be stable (inside the unit circle) or rejected. This procedure, known as pole relocation, is iterated using the estimated poles as starting poles. The solution procedure is based on least squares approximation. Using this method, convergence is often reached within a few iterations.

B. Residue Identification

The residues can be easily solved using (5) since the poles are now identified. To make sure that the transfer function has real-valued coefficients, the linear problem is constructed in such a way that a complex-conjugate pole pair results in a complex-conjugate residue pair.

The main reason for the fast convergence of the ZDVF is that the pole relocation occurs inside the stable unit circle and the initial starting pole relocate a very small distance inside the unit circle as opposed to the s-domain vector-fitting scheme in which the poles travel long distances within the stable left hand side of the s-plane.

IV. SELECTION OF STARTING POLES

A possible way of selecting a starting pole was introduced in [5] and [6] for s-domain vector-fitting. In [5], the poles are selected in a logarithmic or linear scale in the frequency range and complex conjugate pole selection. In ZDVF, the starting poles are selected as complex-conjugate pairs in a linear scale and in the proximity of the unit circle in the stable region. The number of starting poles determines the order of the approximation. The poles are formulated as

$$\{\text{Starting poles}\} = \alpha e^{\pm j\pi\theta_i} \quad (12)$$

For example, α can be chosen 0.9 and θ_i is an angle chosen in equally spaced scale where $0 \leq \theta_i \leq 2\pi$ in complex-conjugate pairs.

V. STATE-SPACE FORMULATION

For a system characterized by scattering matrix $\mathbf{S}(j\omega)$, the input is a forward wave to the ports $\bar{a}(t)$ and the output is a reflected wave from the ports $\bar{b}(t)$ in time domain. Thus, the N-port S-parameter equation is given by

$$\bar{b}(t) = \mathbf{s}(t) * \bar{a}(t) \quad (13)$$

where $*$ is direct convolution. The termination condition is given by

$$\bar{b}(t) = \Gamma \bar{a}(t) + TV_g(t) \quad (14)$$

where Γ is the reflection coefficient matrix, T is the voltage division matrix, and $V_g(t)$ is the source voltage vector.

$$\Gamma = -[I + Z_T Z_0^{-1}]^{-1} [I - Z_T Z_0^{-1}] \quad (15)$$

$$T = [I + Z_T Z_0^{-1}]^{-1} \quad (16)$$

Z_T is the termination impedance matrix and Z_0 is the reference impedance matrix. Once the poles and the residues of $\mathbf{S}(z)$ are obtained, the equivalent state-space model can be formulated for time-domain simulation. Since, the system is assumed to be a linear time-invariant discrete-time system, it can be expressed in a state-space form as

$$x[k+1] = Ax[k] + B\bar{a}[k] \quad (17)$$

$$\bar{b}[k] = Cx[k] + D\bar{u}[k]$$

where $x \in \mathbb{R}^n$ represents the state variables of the system, $\bar{b} \in \mathbb{R}^q$ and $\bar{u} \in \mathbb{R}^p$ are the q -output reflected wave and p -input

forward wave of the system respectively. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$ and $D \in \mathbb{R}^{q \times p}$ are constant matrices for the linear time-invariant system. The state matrix A contains the poles in diagonal and B is a single column matrix that contains all ones. The C matrix contains the residues in a single-row matrix, and D is zero in this formulation. The general solution for the above equation can be achieved by recursive manner. It is obvious that for a given initial condition $x[0]$,

$$x[1] = Ax[0] + B\bar{a}[0]$$

$$x[2] = Ax[1] + B\bar{a}[1] = A^2x[0] + AB\bar{a}[0] + B\bar{a}[1]$$

Let $\Phi[k] = A^k$. Therefore, the general solution becomes [8]

$$x[k] = \Phi[k]x[0] + \sum_{j=0}^{k-1} \Phi[k-1-j]B\bar{a}[j] \quad (18)$$

$$\bar{b}[k] = C\Phi[k]x[0] + \sum_{j=0}^{k-1} C\Phi[k-1-j]B\bar{a}[j] \quad (19)$$

Equation (19) gives the output reflected wave at each port at each time step for the given excitation input. The terminal voltage and current at each port can be easily calculated from the forward and reflected waves.

VI. NUMERICAL EXAMPLES

The scattering-parameters of a 2-port interconnect structure is measured in the frequency range 50 MHz - 20 GHz and the data is used as an example to demonstrate the accuracy and fast convergence of ZDVF. For comparison purposes, the data is approximated using both VF and ZDVF. The initial poles for VF is chosen as proposed in [5] while the initial poles for ZDVF are chosen as discussed in this paper. Table 1 shows the comparison between the two methods for the given data. It shows that ZDVF converge within two iterations with a relatively lower order of approximation and better accuracy compared to the VF method. Since ZDVF converges faster than VF, the overall macromodeling time reduces significantly. Figures 1 and 2 show results for the magnitude and phase of S_{21} for the measured and the approximated values using ZDVF. Figure 3 shows the pole locations in z-plane obtained using ZDVF after two iterations. From the pole plot, the optimal poles are located close to the unit circle in the stable region. This shows choosing starting poles close to the unit circle as proposed gives a faster convergence.

Figure 4 shows a comparison of the condition number of the pole identification between the VF and ZDVF methods. It shows that ZDVF constructs a better conditioned matrix to identify the poles of the system. Figure 5 shows a comparison of the root-mean squared (RMS) error between VF and ZDVF over 15 iterations for both method. For ZDVF, the error drops rapidly at the first few iterations and remains constant at high iterations. It shows also ZDVF gives better accuracy even for a lower order of approximation.

Figure 6 shows the transient simulation results for a pulse input excitation with a matched termination condition at port 1

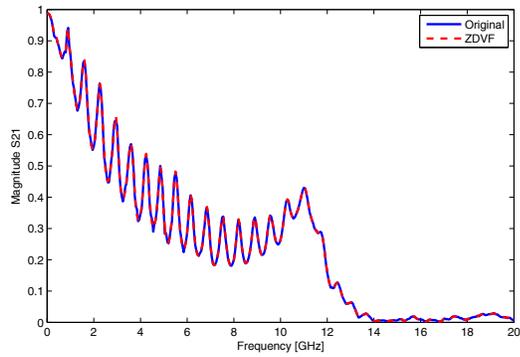


Fig. 1. Magnitude comparison of S_{21} of the measured data and ZDVF

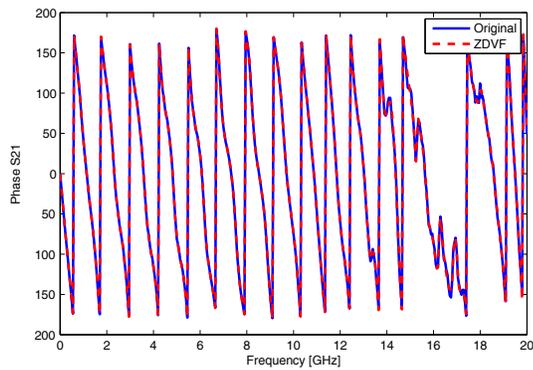


Fig. 2. Phase comparison of S_{21} of the measured data and ZDVF

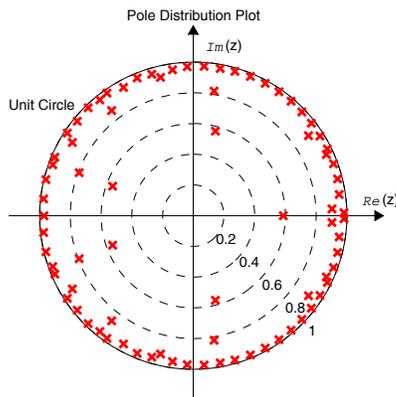


Fig. 3. Plot of poles location in z-plane

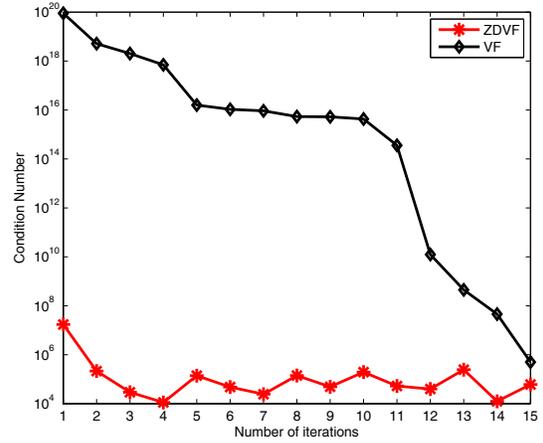


Fig. 4. Condition number of VF versus ZDVF

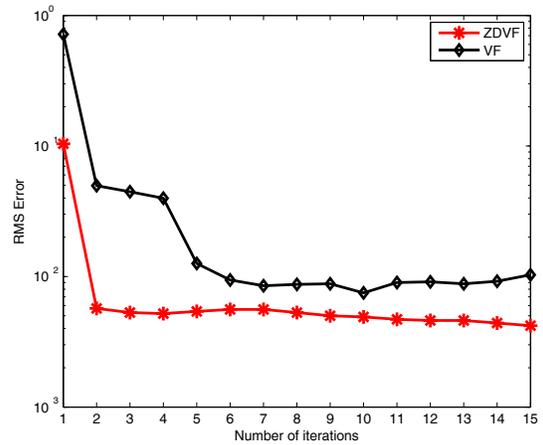


Fig. 5. RMS Error VF versus ZDVF

and open at port 2. It shows the transient response at each port using a direct convolution technique [10] and ZDVF. Clearly, the result matches very well.

	VF	ZDVF
Number of Iteration	5	2
Order of Approximation	96	80
RMS error	0.0291	0.0059

TABLE I
COMPARISON BETWEEN ZDVF AND VF METHOD

The fast convergence of ZDVF is dependent on the starting poles location inside the unit circle. To illustrate, ZDVF simulation is performed by changing the starting pole locations α . Figure 7 shows how the condition number at each iteration is changing for different values of α . For a small value of α near to the center of the unit circle, the condition number is large

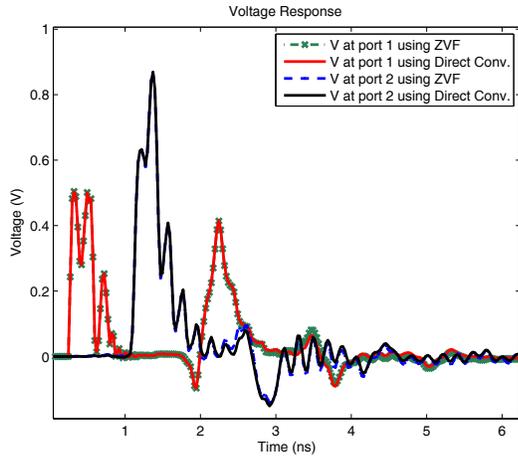


Fig. 6. Time-domain comparison between direct-convolution and ZDVF

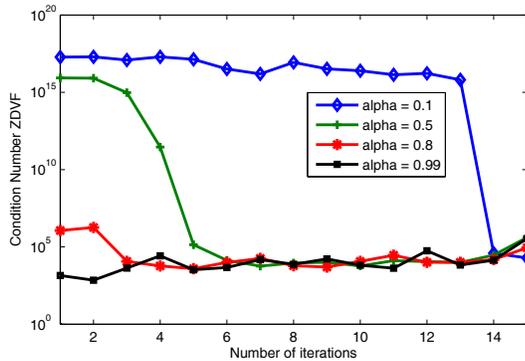


Fig. 7. Condition Number of ZDVF as α change

at each iterations which results a stability problem. Figure 8 shows how the RMS error at each iteration is changing for different values of α . Starting poles selected near the unit circle has a faster convergence compared to those poles selected near the center.

VII. CONCLUSION

The z-domain vector-fitting method (ZDVF) has been proposed and tested to improve the convergence and numerical stability of vector-fitting method to construct broadband macromodel over wide frequency range. The pole-relocation is confined within a unit-circle in the z-plane and therefore a faster convergence is achieved. In addition, selecting initial poles close to the unit-circle provides an even faster convergence. The convergence rates of the vector fitting method and the proposed ZDVF method are compared. The ZDVF method exhibits a rapid decrease in the error and stabilizes at high iterations.

ACKNOWLEDGMENT

This project is partially supported by Cadence Design System.

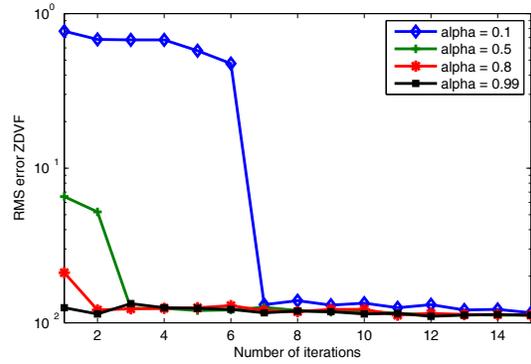


Fig. 8. RMS Error of ZDVF as α change

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